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TURBULENT VELOCITY DISTRIBUTIONS

IN THE UPPER ATMOSPHERE

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William B. Moseley

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TURBULENT VELOCITY DISTRIBUTIONS

IN THE UPPER ATMOSPHERE

Approved:

Chairman

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Dedicated to those whose love and understanding

made this possible:

my wife, Sandra;

my children, Michelle, Theresa, and Kimberly;

and our family relations both living and deceased.

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GLOSSARY OF FREQUENTLY USED SYMBOLS

Symbol	Definition
$\overleftrightarrow{A}$	Tensor which is inverse of matrix of variances and covariances
A_i	The negative of the average value of the i th component of the spatially-dependent force
A_{ijk}	Coefficients of trivariate Hermite Polynomials
a_{ij}	An element of tensor $\overleftrightarrow{A}$ given in equation A.88
b_{ij}	An element of the cofactor matrix of $\overleftrightarrow{A}$
$C(t)$	Characteristic function
D_i	Coefficients of the univariate Hermite polynomials
F_i	Spatially-dependent forces
$F(x)$	Cumulative distribution function
f	Probability density function, abbreviated p.d.f.
f_x	Spatially-dependent probability density function
f_v	Velocity-dependent probability density function
f_{v0}	Unperturbed solution for velocity-dependent p.d.f.
$f_1(v_1)$	One-dimensional p.d.f. for parallel component of turbulent velocity
$f_2(v_2)$	One-dimensional p.d.f. for perpendicular component of turbulent velocity
$f_3(v_3)$	One-dimensional p.d.f. for vertical component of turbulent velocity
$f(x,\theta)$	Conditional probability density function
G_{ij}	Complementary bivariate Hermite polynomials
G_{ijk}	Complementary trivariate Hermite polynomials

Glossary (Cont'd.)

Symbol	
g	Gravitational acceleration
H_i	Univariate Hermite polynomials
H_{ij}	Bivariate Hermite polynomials
H_{ijk}	Trivariate Hermite polynomials
h_i	Arbitrary constants
K	Constant defined by equation A.89
k_i	Constant parameters defined in equation F.44, F.45, and F.46.
k_{qrs}	k-statistics defined by equation A.26
L	Scale parameter Likelihood function
M-test	Analysis of variance test
m_{ijk}	Trivariate moment of the sample about the mean
m'_{ijk}	Trivariate moment of the sample about the origin
N	Sample size
n	Number of observations in sample
P	Instantaneous pressure
P_o	Mean pressure
p	Fluctuating pressure
S_{ijk}	A power sum over the observations in the sample
T	Instantaneous temperature
T_o	Mean temperature
T_L	Launch time for rocket
t	Time

Glossary (Cont'd.)

Symbol	
$t_{\langle v \rangle}$, t_{γ_1}, t_{β_2}	Statistics used in T-tests
U_i	Component of instantaneous wind velocity
V_i	Component of mean wind
$\frac{\partial V_i}{\partial x_j}$	Shear tensor
v_i	Component of turbulent velocity
$\langle v \rangle$	Mean turbulent velocity
w_i	Average value of i th component of the shear vector
x_i	Component of spatial coordinate i th value of variate in a sample
β	Non-dimensionalizing constant
β_2	Flatness factor
γ	Intermittency factor
γ_1	Skewness
Δ	Determinant of $\underset{\leftrightarrow}{A}$
δ_{ij}	Kronecker's delta
ζ	Dimensionless perturbation parameter
θ	Temperature fluctuation
κ_{ijk}	Cumulants defined by equation A.25
λ	Scale parameter

Glossary (Cont'd.)

Symbol	
μ_{ij}	Dimensionless tensor which couples velocity components
μ_{ijk}	Trivariate moment of the population about the mean
μ'_{ijk}	Trivariate moment of the population about the origin
ν	Kinematic viscosity Scale parameter
ξ_i	Variables defined in equations F.50, F.51, and F.52
ρ_o	Mean density
ρ_{ij}	Correlation coefficient of population
σ	Standard deviation Scale parameter
σ_i	Standard deviation of the i th component of the turbulent velocity
ϕ	Angle of rotation between geographic system and shear system Quadratic form used in generating Hermite polynomials
ϕ_i	i th order perturbation correction in the dimensionless velocity-dependent probability density
χ^2-test	Chi-squared goodness-of-fit test
ψ	Quadratic form used in generating complementary Hermite polynomials
ψ_i	i th order perturbation correction in the dimensional velocity-dependent p.d.f.
Ω	Spatially-dependent scalar potential
<>	Expectation value or mean value
^	Denotes dimensionless quantity
$\leftrightarrow$	Denotes a matrix
∿	Denotes a vector

SUMMARY

Observations of turbulent velocity components in the 90 to 110 km region of the upper atmosphere were made on twelve chemical release clouds. The probability density function, p.d.f., was computed for each of the velocity components in the geographic coordinate system and observed to be non-Gaussian. Several moments and parameters such as the skewness and β_2, the flatness factor, were computed for the observed p.d.f.'s. The non-Gaussian character of the p.d.f. is consistently exhibited through high values of β_2. The application of t-tests indicated that the values of β_2 are significantly different at the 1% level from the value for a Gaussian p.d.f. It is shown that the high values of β_2 cannot occur as a result of simultaneous sampling of Gaussian turbulent velocities and Gaussian errors. It is also demonstrated that it is highly improbable (at the 1% level) that the non-Gaussian character is due to an intermittent observation in the first case of Gaussian turbulent velocities and Gaussian errors or in the second case of Gaussian errors alone.

The observations were converted to the shear system which has axes along the direction of the shear in the mean wind, perpendicular to the shear direction but in the horizontal plane, and along the vertical direction. The p.d.f.'s for each of the turbulent velocities were computed for all of the data grouped together and then for portions of the data belonging to categories such as particular altitude intervals and shear magnitude intervals. The flatness factor was consistently

non-Gaussian at the 1% level throughout the various categories of the data. The skewness is shown to be significantly non-Gaussian at values of the shear magnitude > 17 m/sec/km.

Confirmation of the non-Gaussian character of the p.d.f.'s was obtained by the use of χ^2 goodness-of-fit tests. The method of analysis of variance indicates that the shear vector definitely influences the turbulent velocities and thus contributes to the non-Gaussian character of the p.d.f. Further analysis of variance shows that over the short lifetime of the releases the p.d.f. was relatively stationary.

Utilization of the experimental information, the equation of motion for fluctuating quantities, and the continuity equation in a development similar to that found in classical statistical mechanics leads to a differential equation for the velocity-dependent probability density function f_v. The differential equation for f_v is solved through a perturbation method of solution. The unperturbed solution is a trivariate Gaussian p.d.f., and the solution to the first-order perturbation differential equation is a series of trivariate Hermite polynomials. The differential equation places "constraint" equations on the A_{ijk} coefficients of the Hermite polynomials. These "constraint" equations provide, in actuality, relations between the moments of the distribution.

The solution f_v which includes the first-order perturbation solution gives substantially better agreement with the observed data than does a Gaussian p.d.f. This improved agreement is demonstrated through truncation relations which provide relations between moments of different orders and through the computation of the one-dimensional p.d.f.'s.

CHAPTER I

INTRODUCTION

Definition of Turbulence and Basic Concepts

Turbulent motion of a fluid is characterized by several properties. The equations of motion are three dimensional, non-linear equations. The motion must be rotational and dissipative, diffusive, a continuum phenomenon, such that mechanical energy is converted into internal energy through a cascade of eddies of diminishing sizes, and such that the various quantities exhibit a random variation with respect to time and space coordinates.

Thus the velocity components have a random variation; but, because of the continuum nature of turbulence, the random variation versus time or space gives rise to an irregular yet continuous velocity field which has derivatives in both time and space. Statistically distinct averages can be discerned in the velocity field, and significant correlations can be obtained over finite intervals of space or time.

Regardless of how carefully the conditions of a laboratory experiment are reproduced, the turbulent velocity field cannot be reproduced in detail. Although a complete solution of the equations of motion for turbulent flow has not been obtained, and although uniqueness of the solution for a given set of conditions has not been demonstrated, uniqueness of the physical problem seems intuitively likely. However the solution appears to be so sensitive to minute changes in the conditions that one can never know these precisely enough to predict the detailed structures of the flow. For this reason and because for most purposes one does not want or need information on the details of the flow, one relies on statistical methods of description. In fact, Batchelor [1953] has stated:

> There is clearly a close correspondence between the subjects of turbulence and statistical mechanics. Statistical mechanics is concerned with the properties of a large number of particles whose motion conforms to certain collision laws, whereas turbulence is concerned with a continuum whose motion conforms to "the continuity equation and the Navier-Stokes equation of motion".

The equation of continuity for turbulent flow in an incompressible fluid is

$$\frac{\partial v_i}{\partial x_i} = 0 \tag{1.1}$$

where v_i is the i th component of the turbulent velocity and where the Einstein summation convention is being used. The Navier-Stokes equation of fluctuating motion is

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial V_i}{\partial x_j} + V_j \frac{\partial v_i}{\partial x_j} + v_j \frac{\partial v_i}{\partial x_j} - < v_j \frac{\partial v_i}{\partial x_j} >$$

$$= - \frac{1}{\rho_0} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 v_i}{\partial x_j \partial x_j} + \frac{g}{T_0} \theta \delta_{3i} \tag{1.2}$$

where V_i is the i th component of the mean wind, t denotes time, x_i is distance along the i th coordinate axis, ρ_0 is the mean density, p is the fluctuating pressure, ν is the kinematic viscosity, g is the magnitude of the gravitational acceleration, θ is the temperature fluctuation, T_0 is the mean temperature, δ_{ij} is Kronecker's delta, and $< >$ denotes averages. Explanations of the above equations may be found in Lumley [1964]. It should be noted that, when dealing with a turbulent medium, the instantaneous values of a quantity are composed of a mean value upon which is superimposed a fluctuating value whose average is zero. Thus the instantaneous values of velocity, temperature, and pressure are given by

$$U_i = V_i + v_i \tag{1.3A}$$

$$T = T_0 + \theta \tag{1.3B}$$

$$P = P_0 + p \tag{1.3C}$$

where $V_i = \langle U_i \rangle$, $T_0 = \langle T \rangle$, and $P_0 = \langle P \rangle$ and, by definition

$$\langle v_i \rangle = \langle \theta \rangle = \langle p \rangle = 0. \tag{1.4}$$

One must be careful in following the analogy between turbulence and statistical mechanics because the analogy is imperfect in two important points. First, at any instant the motion of a gas molecule affects appreciably the motion of at most one other molecule; whereas in turbulent motion, the motion at any point influences the motion at many other distant points. Secondly, the turbulent motion requires a continuous supply of energy to maintain it, which is obtained from the interaction of the mean flow and the turbulent stresses. More detailed discussions may be found in Batchelor [1953] and Townsend [1956].

There are various methods of categorizing turbulent flows; a discussion of some of these is given by Hinze [1959]. One system of classification is based upon the way in which the turbulence is generated. Turbulence can be generated by frictional forces at fixed boundaries or by the flow of layers of fluid with different velocities past or over one another or by the existence of an unstable stratification with respect to temperature or by the flow of a uniform stream of fluid through a regular grid of bars, held at right angles to the mean motion. For purposes of identification it will be convenient to designate the types of turbulence mentioned above as wall, shear turbulence; free, shear turbulence; instability turbulence, and grid turbulence respectively. It should be noted that, as regards the mechanism of production, wall, shear turbulence and grid turbulence are "close cousins" to free, shear turbulence when the three types are compared to instability turbulence.

Another system of classification is based upon the properties of the flow field. If the turbulence has quantitatively the same average structure in all portions of the flow field, the turbulence is called homogeneous. If the statistical features of the turbulence have no preference for any direction, the turbulence is isotropic. When the mean velocity of the flow field has a gradient, and consequently an average shear stress occurs, the turbulence is nonisotropic. For the case of a constant average shear, the turbulence is called homologous turbulence. Psuedo turbulence refers to the hypothetical case of a flow field with a regular pattern that shows a distinct constant periodicity in time and space.

The turbulence occurring in the region of the upper atmosphere investigated here is extremely complicated and could be designated as nonhomogeneous, nonisotropic, free, shear turbulence.

Methods and Difficulties of Past Research

The principal difficulties encountered in attempting to solve the problems of turbulence arise from the three-dimensional character of the velocity field, the non-linearity of the equation of motion, and the random nature of the velocity with the subsequent need for statistical methods. Because of these difficulties, research has concentrated on the simpler forms of turbulence. Due to its relative simplicity, homogeneous isotropic turbulence has been studied more than any other form. The geometrical and kinematic relations involved in this type of turbulence have been investigated through correlation and energy spectrum functions, and the dynamic behavior of these correlation and energy spectrum functions. Up to now it has been possible to obtain a nearly complete picture of the formal geometrical relations for homogeneous isotropic turbulence in an incompressible fluid. However, it has not been possible to obtain general, complete solutions of the differential equations involved for even this simple type of turbulence. Nor are complete solutions known for the various correlation and energy spectrum functions, nor for their changes with time.

Axisymmetrical turbulence, which is second in simplicity to isotropic turbulence, has been investigated by several researchers. The kinematics of this type of turbulence has been treated almost as thoroughly as was the kinematics of isotropic turbulence. The method of investigation was the same. However, the axisymmetrical turbulence that was treated involved an assumed uniformity of the main flow and the absence of mean turbulence shear stresses.

Recently, the beginnings of the same method of analysis have been applied to free, shear turbulence by Justus [1968]. Townsend [1956] presents a few types of psuedo-turbulent flows which simulate real turbulent fields and thus are useful for investigating various characteristics of real turbulent flows.

In the past, it was expected that a plot of the probability density of amplitudes of turbulent velocities for isotropic turbulence would be a Gaussian distribution. Experimental measurements have verified this. For free, shear turbulence the distribution was expected to approximate a Gaussian distribution with the rather vague generalization that the distribution would be more or less skew.

Purposes of Current Research

As can be surmised, relatively little is known about free, shear turbulence. The purpose of this current work is to provide a theory which leads ultimately to the form of the velocity probability density distribution of homologous, free, shear turbulence in the height region around 100 km. A knowledge of this distribution could eventually lead to a somewhat different method of attack on the problems of turbulence.

With regard to general research in upper atmospheric physics, the purposes are indeed numerous and the need is great. If a survey were taken of all physicists and meteorologists, the list of reasons for upper atmospheric research would probably include hundreds of different items. Obviously, only a very few can be mentioned here. Perhaps the research most vital to the national defense is a program in which artificial clouds of ions are released into the atmosphere by rockets. Researchers hope to gain some knowledge about the effects of nuclear detonations on communication and tracking abilities. Another phase of this research deals with the feasibility of reflecting signals from these ion clouds and thereby maintaining communications between distant points during ionic storms.

Still another problem involves the determination of density profiles and fluctuations of the atmosphere. The atmosphere can be divided into two reasonably distinct regions according to its composition: the homosphere, which extends up to about 100 km and in which the composition and average molecular weight are essentially constant; and the heterosphere, which continues upward from about 100 km and in which the composition varies with altitude. In the homosphere local turbulence and various winds induce a continuous mixing of the gaseous components of the atmosphere thereby providing the constant composition. In the heterosphere, diffusive separation occurs so that the molecules are distributed vertically in accordance with their masses. Changes in the ceiling of the turbulence and wind-induced homosphere may cause the heterosphere to react in a similar but amplified manner and thus produce density variations at satellite altitudes. Another reason for this type of research is that a knowledge of the ambient density would allow calculation of forces exerted on rockets and space vehicles as they pass through the atmosphere.

CHAPTER II

EXPERIMENTAL OBSERVATIONS

Procedure

The height region under investigation by the present study consists of a range of altitudes from 90 to 110 km. This region is above the ceiling of balloon observations; the upper half of the region is above most meteor observations and below the perigee of satellites. Thus rocket probes which release chemi-luminescent clouds into the atmosphere are used in conjunction with an optical tracking network on the ground.

The total winds in the atmosphere consist of a superposition of prevailing winds, waves such as periodic tidal waves, irregular winds such as gravity waves and turbulent components. The mean wind profile V(z) which extends over a wide altitude range is provided by photographic tracking of the chemical release clouds [Justus et al., 1964a, 1964b]. These mean winds are averages of the motion over the cloud lifetime, which is usually between five and ten minutes. Quite often protuberances in the form of nearly spherical globules appear on these clouds below about 106 km. Globules are defined here to be small-scale structure superimposed on the gross cloud features. Occasionally large-scale, near spherical blobs, which constitute the entire cross-section of trail releases, are observed as high as 120 km. However the large-scale blobs are attributed to anomalies of the release and not associated with ambient turbulence. Figure 1 shows a trail release that illustrates the globular small-scale structure particularly well between 98 and 100 km. The numbers refer to km of altitude, and the arrows point to the approximate position on the cloud at which the altitude is attained. Identifiable features or points on the cloud, such as globules, can be tracked individually over the lifetime of the cloud. Position determinations of these features at various times throughout the duration of the cloud can then be used to obtain a set of time-varying winds for each globule or feature observed. Figure 2 shows the observed position coordinates versus time for the center point of a typical globule. Instantaneous wind components are the slopes of the lines connecting consecutive points. Turbulent winds for the various times are computed by subtracting the mean wind determined by the motion of the trail centerline from the instantaneous winds. The globule shown in Figure 2 exhibits substantial irregular motion in the north-south and vertical directions with approximately constant east-west motion. Figure 3 shows the north-south component of a typical mean wind profile and near 105 km indicates the instantaneous N-S winds and the N-S component of the turbulent winds at time T_L + 213 sec, where T_L is the launch time for the rocket.

Because of the relatively short lifetime of these clouds, periodic tidal wave and large time scale gravity wave and/or large time scale turbulent components would be considered as part of the mean winds. However, results (Justus [1965]) indicate that the scales of turbulence observable by chemical release tracking

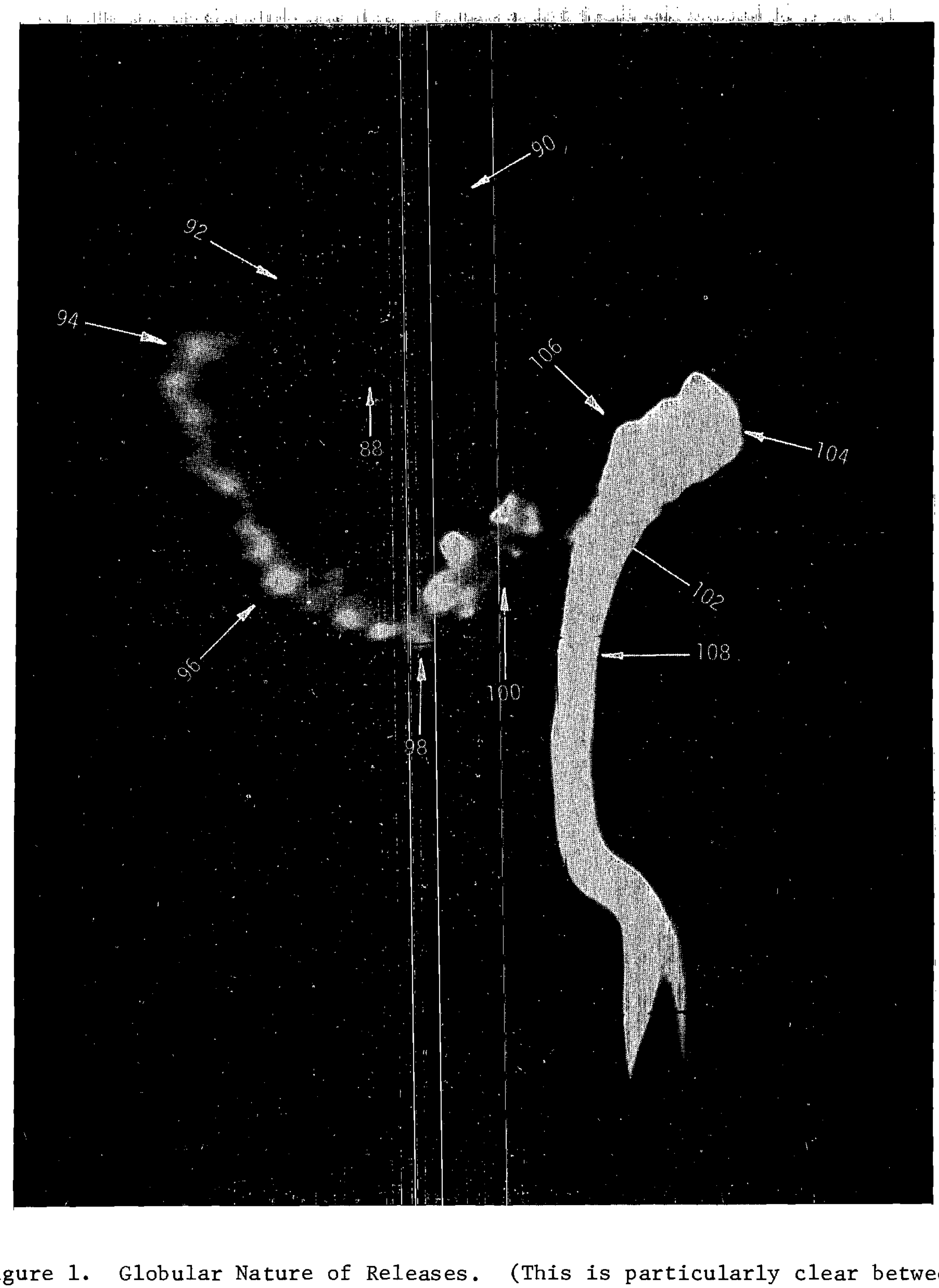

Figure 1. Globular Nature of Releases. (This is particularly clear between 98 and 100 km. The numbers refer to km of altitude and the arrows point to the approximate point on the cloud at which the altitude is attained.)

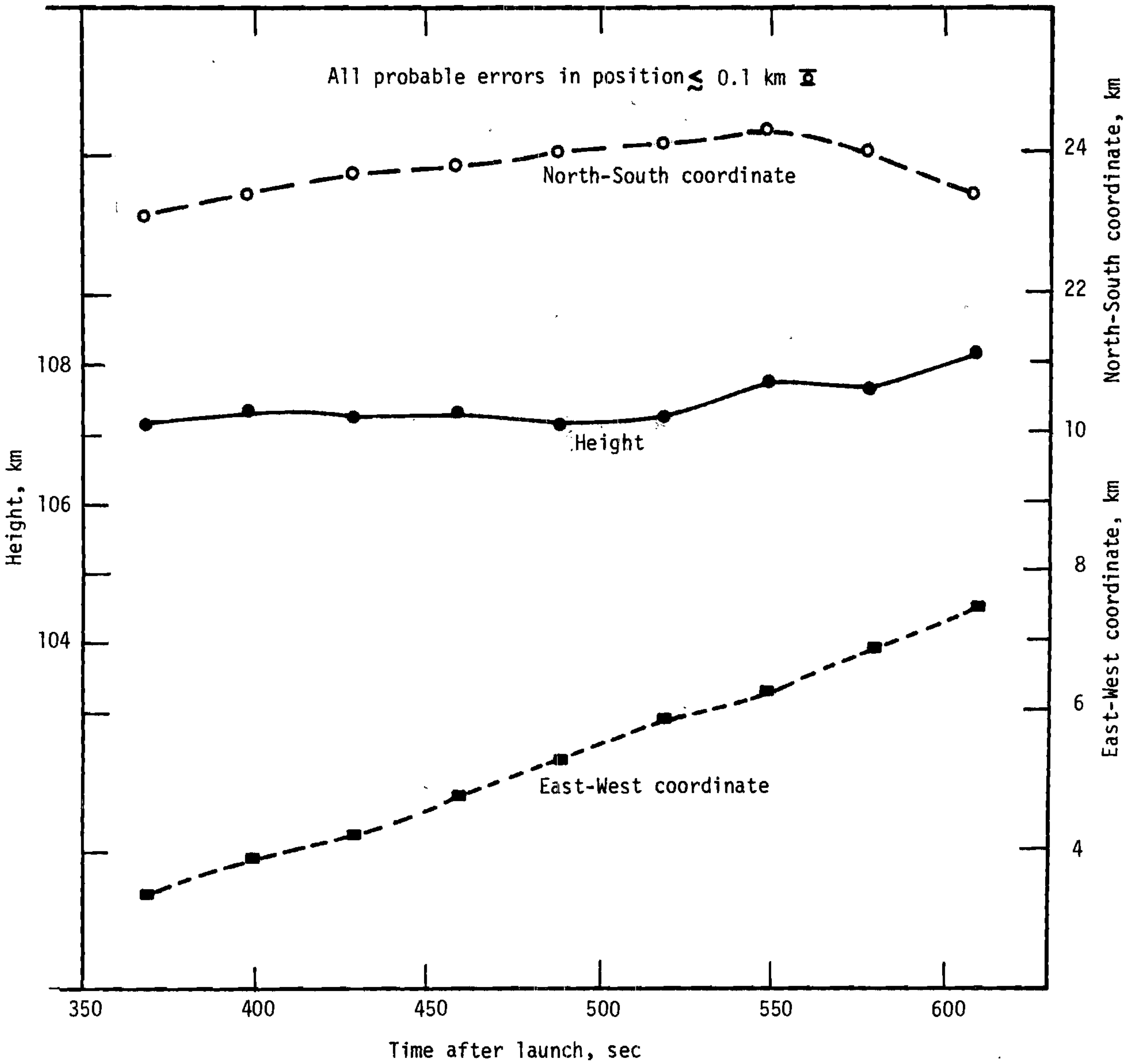

Figure 2. Coordinate Components Versus Time for the Center Point of a Globule on a Cesium Trail Release.

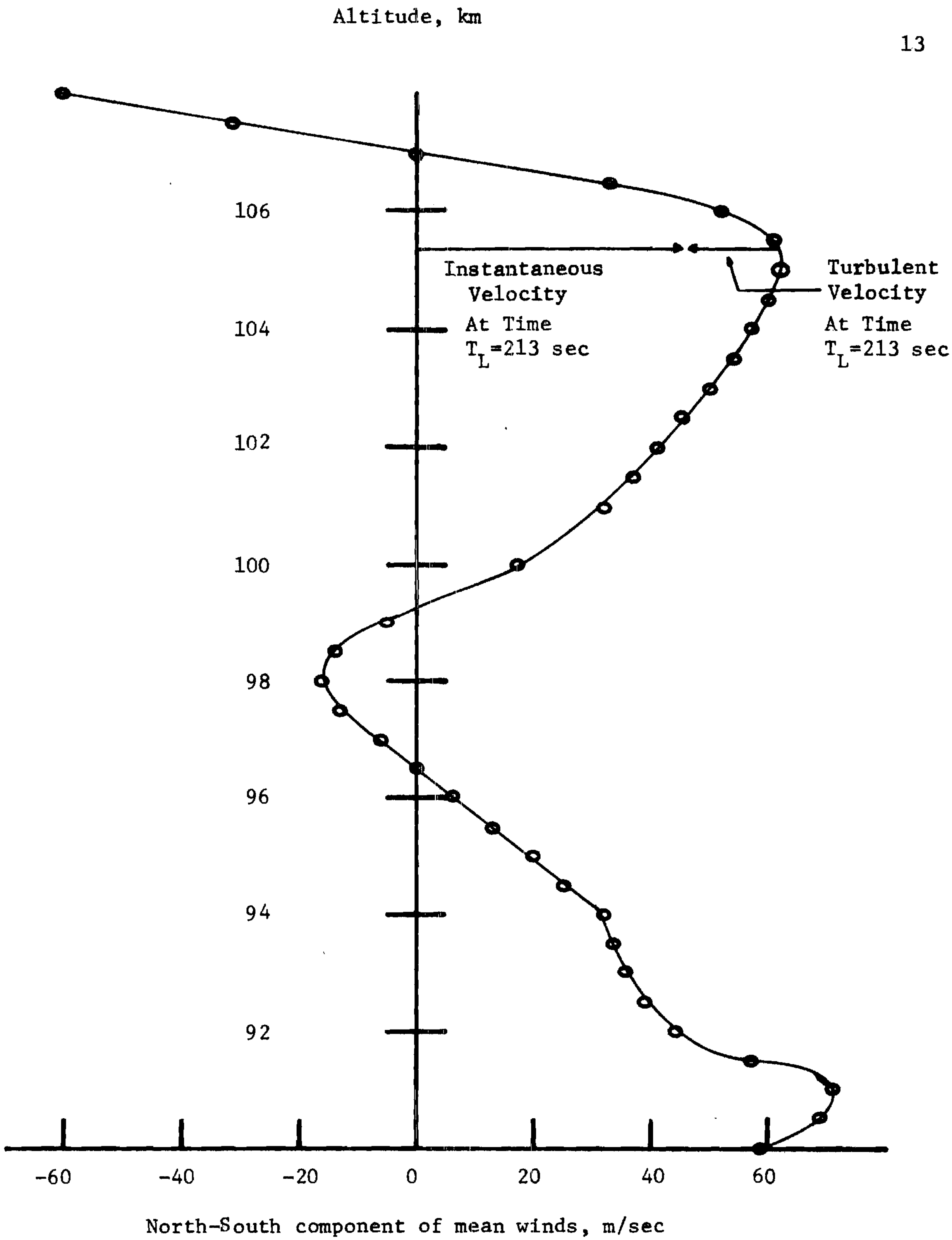

Figure 3. The North-South Component of the Mean Wind Velocity Profile Versus Altitude for a Typical Release Cloud. (The instantaneous north-south wind component and the north-south turbulent wind component are shown at 105.35 km for a typical time [Launch plus 213 sec].)

constitute most, if not all, of the larger scales in the turbulent spectrum.

At this point the reader should familiarize himself with the contents of Appendix A, since portions of it will be used throughout the remainder of the work.

Initial Data in Geographic Coordinate System

The most convenient coordinate system for measuring the various types of winds is a geographic coordinate system. In this system, the axes are oriented as follows: the EW axis extends from the origin to the East; the NS axis extends from the origin to the North; and the z axis extends vertically upward from the origin. The origin is at the sea-level intersection of a reference latitude and longitude. Thus, northward, eastward, and vertical turbulent velocity components were determined for twelve chemical release clouds. As is demonstrated in Appendix B, the turbulent winds can be determined to within an accuracy of about 3 m/sec. These clouds gave rise to a total of 2308 individual observations of the turbulent velocity components. In all cases the experimental p.d.f.'s for the velocity components were computed by counting the number of velocity component observations that have values within 1-m/sec. intervals. This count was then divided by the total number of observations of the sample to give a frequency (in percent of the total observations) for that particular velocity interval. Figures 4, 5, and 6 show the computed p.d.f.'s for the northward, eastward, and vertical turbulent velocity components in the 90-110 km region. All three figures were based on the 2308

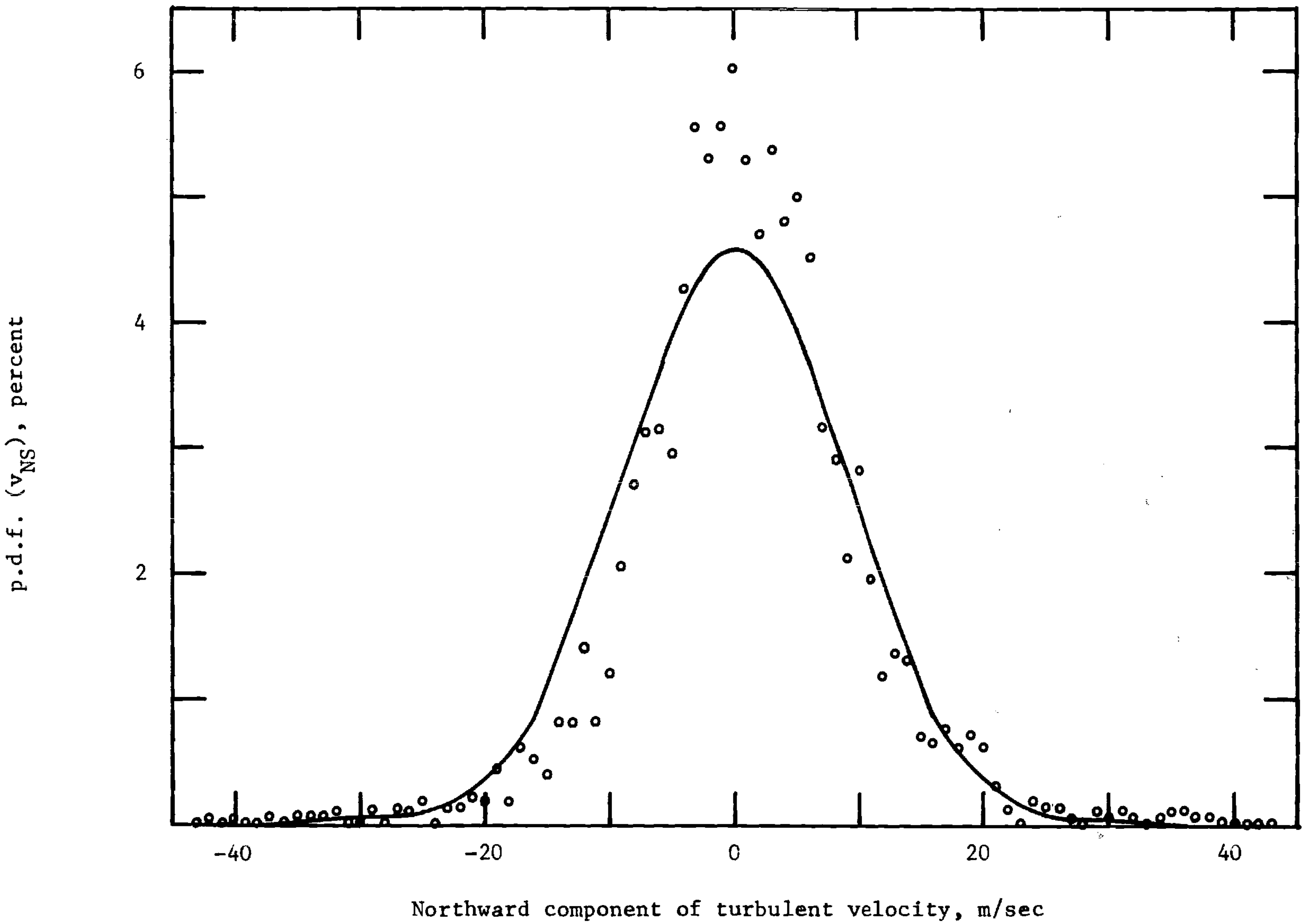

Figure 4. The p.d.f. for the Northward Component of the Turbulent Velocity in the 90-110 km Region. (The smooth curve is a Gaussian distribution with the same standard deviation and the same mean as the observed distribution.)

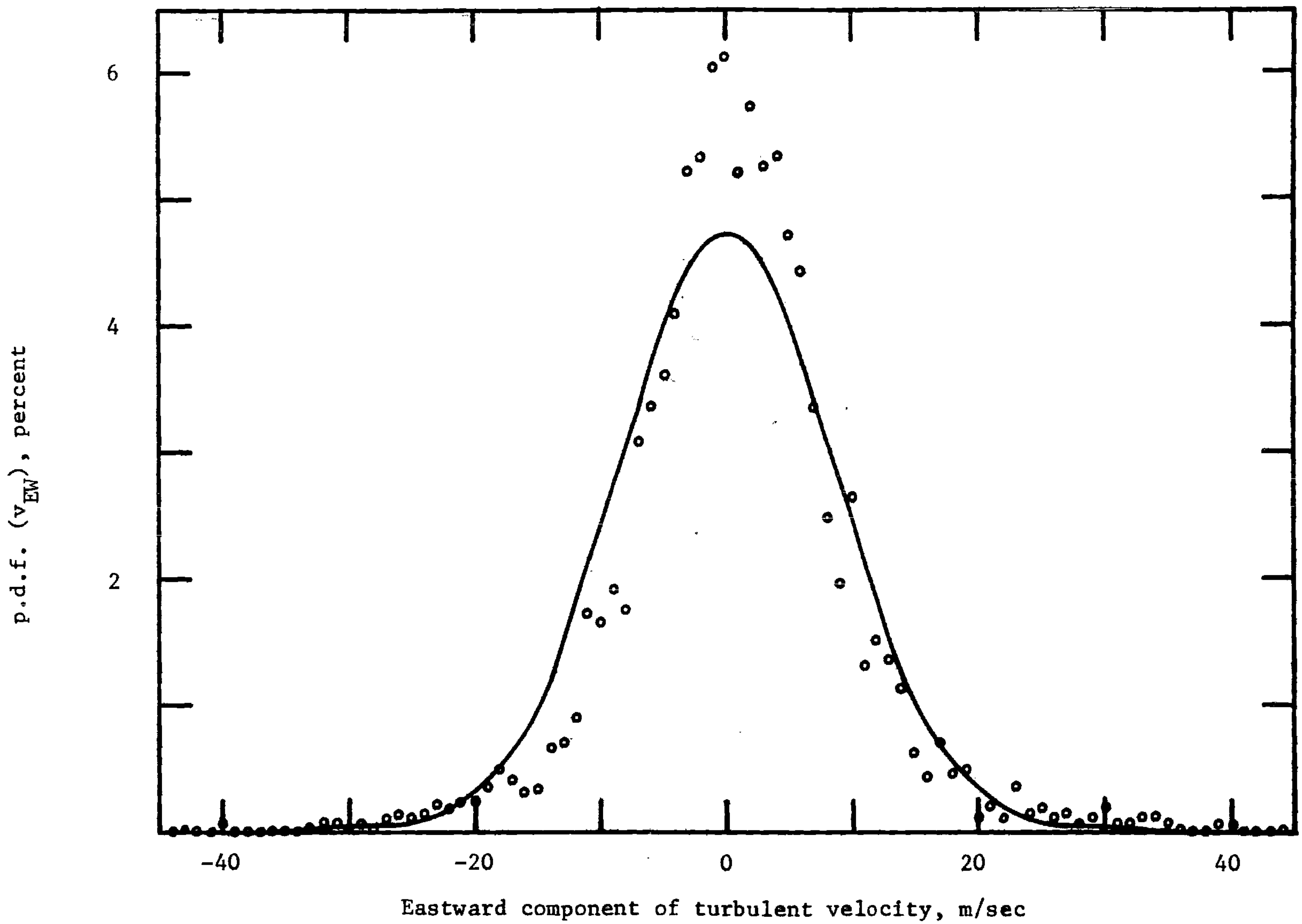

Figure 5. The p.d.f. for the Eastward Component of the Turbulent Velocity in the 90-110 km Region. (The smooth curve is a Gaussian distribution with the same standard deviation and the same mean as the observed distribution.)

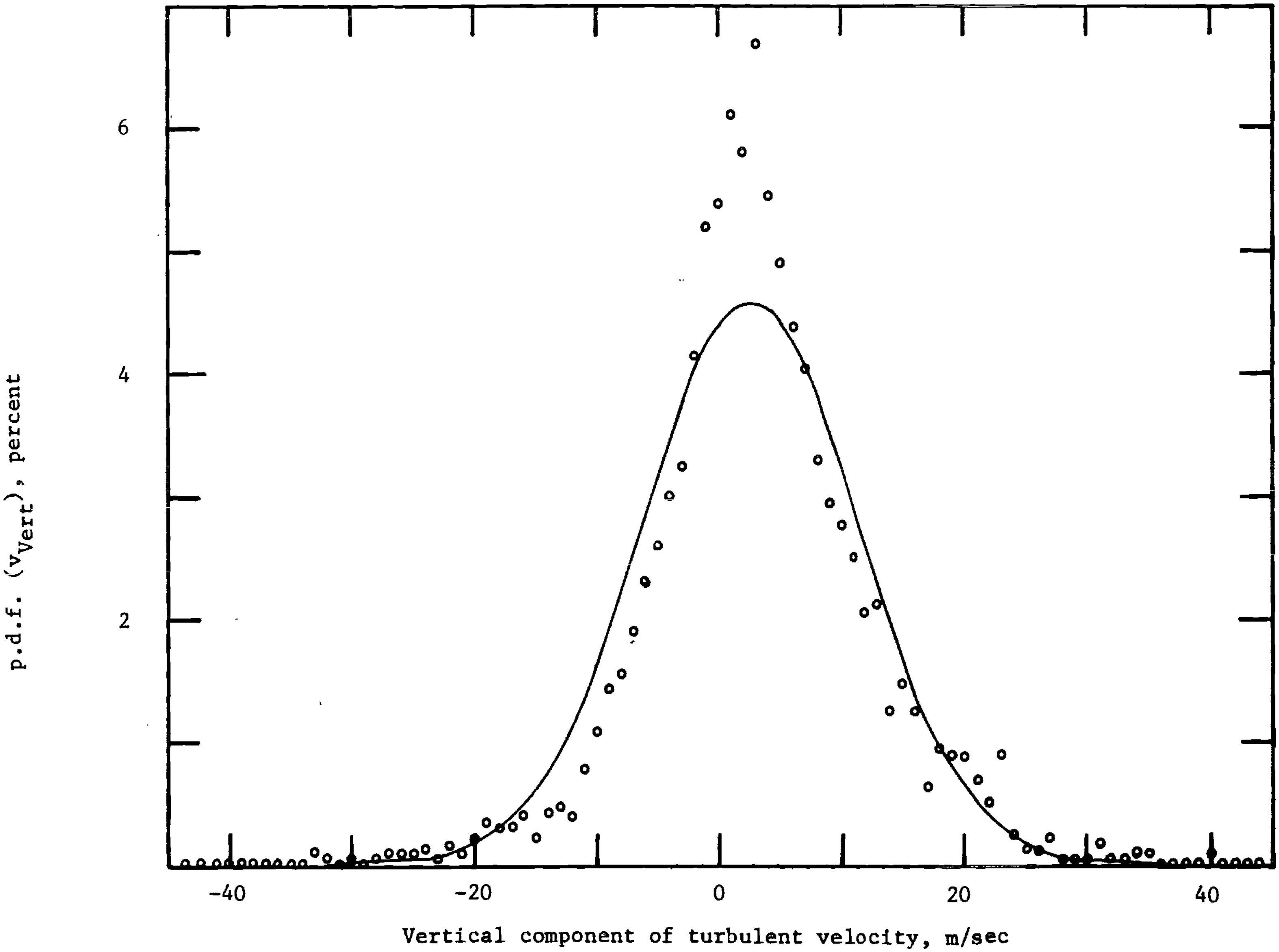

Figure 6. The p.d.f. for the Vertical Component of the Turbulent Velocity in the 90-110 km Region. (The smooth curve is a Gaussian distribution with the same standard deviation and the same mean as the observed distribution.)

individual observations of each of the turbulent velocity components. Superimposed on each figure is a smooth curve which represents a Gaussian distribution with the same standard deviation and the same mean as the observed distribution. It becomes immediately obvious that the computed p.d.f.'s are not Gaussian; instead they have high central peaks and broad tails (a certain indication of a high flatness factor β_2). Table 1 shows the calculated values of the standard deviation σ, the mean turbulent velocity $< v >$, the skewness γ_1, and the flatness factor β_2 for the three components. The results substantiate the figures in that high β_2's are obtained. Next, t tests were performed on each of the values for $< v >$, γ_1, and β_2, where the statistics were

$$t_{<v>} = \frac{(<v> - 0)}{\sqrt{\text{var } (<v>)}} = N^{1/2} \frac{(<v>)}{\sigma} \tag{2.1}$$

$$t_{\gamma_1} \doteq \frac{(\gamma_1 - 0)}{\sqrt{\text{var } \gamma_1}} \tag{2.2}$$

$$t_{\beta_2} = \frac{(\beta_2 - 3)}{\sqrt{\text{var } \beta_2}} \tag{2.3}$$

The results of the statistical tests are also shown in Table 1 where an asterisk (*) indicates a highly significant difference between the

Table 1. Parameters of the Observed Probability Density Functions for the Three Turbulent Velocity Components in the Geographic System. (Asterisks (*) indicate values that are very significantly different from the expected value for a Gaussian distribution. 'Highly significant difference' means better than 1% significance level based on a t test.)

Parameter	Expected Gaussian Value	Northward Velocity Component	Eastward Velocity Component	Vertical Velocity Component
σ,m/sec	σ	8.68	8.42	8.69
$\langle v \rangle$,m/sec	0	0.33	0.31	2.40*
$\gamma_1=\sqrt{\beta_1}$	0	0.04	0.22*	0.05
$\beta_2 = \gamma_2+3$	3	5.30*	5.24*	4.49*

observed and predicted values (highly significant implies a significance level of better than 1%). The p.d.f.'s of all three velocity components are characterized by highly significant values of β_2. In Appendix D, it is proven that these large values of β_2, which exhibit the non-Gaussian character of the p.d.f.'s, could not have arisen through the simultaneous sampling of Gaussian velocities and Gaussian errors. Also, in Appendix E, the improbability is demonstrated that this non-Gaussian character of the p.d.f.'s could have been due to intermittent observation in the first case of Gaussian turbulent velocities plus Gaussian errors or in the alternative case of Gaussian errors alone. Therefore the high values of β_2 do indeed exhibit a non-Gaussian characteristic of the turbulent velocities alone.

The mean vertical velocity is highly significantly different from the expected zero value. This significant mean vertical velocity is very likely due to the fact that chemical clouds usually exhibit an initial upward motion, and although an attempt was made to eliminate this anomalous initial vertical velocity from the data, it appears that a portion of this motion is still giving the vertical velocity data a bias toward positive vertical motion. The anomalous initial upward motion is often large in magnitude (>25m/sec), but it usually 'damps out' within one or two minutes after the release of the chemical cloud. Therefore, to eliminate most of the anomalous early upward motion, all observations with vertical winds larger than 25 m/sec at times less than release plus two minutes were removed from consideration. Regardless of the above discussion, high values or non-zero values for the mean of the turbulent velocity components will not affect results based on higher order moments about the mean. Note also that some results (to be found later in this work) are based on the use of cumulants of order two or greater and these cumulants are independent of the values of the mean turbulent velocity. The expected Gaussian values for all moments or parameters of order two or greater were also based on moments about the mean or cumulants. The only significance of a non-zero value for the mean turbulent velocity in the case of horizontal winds is that it gives some indication of the accuracy of the determination of the mean wind shown in the mean wind profile; and in the case of vertical winds, it indicates the degree to which the above mentioned method was successful in eliminating the anomalous upward motion.

Table 1 also shows that the eastward velocity component p.d.f. has a highly significant skewness. Yet there is nothing physically unique about the eastward axis. It was thought that perhaps the grouping together of all of the data from the different altitudes had misleadingly suppressed the skewness of the northward component through compensating positive and negative skewnesses at different altitudes. To remedy this, the turbulent velocity data were divided into four 5-km altitude intervals, and the p.d.f.'s for each component in each height range were computed.

Figures 7, 8, 9 and 10 show the p.d.f.'s for the eastward component of the turbulent velocity for the altitude intervals 90-95 km, 95-100 km, 100-105 km, and 105-110 km respectively. The smooth curve on each figure represents a Gaussian p.d.f. with the mean and σ appropriate to each figure's data. Since the figures for the northward

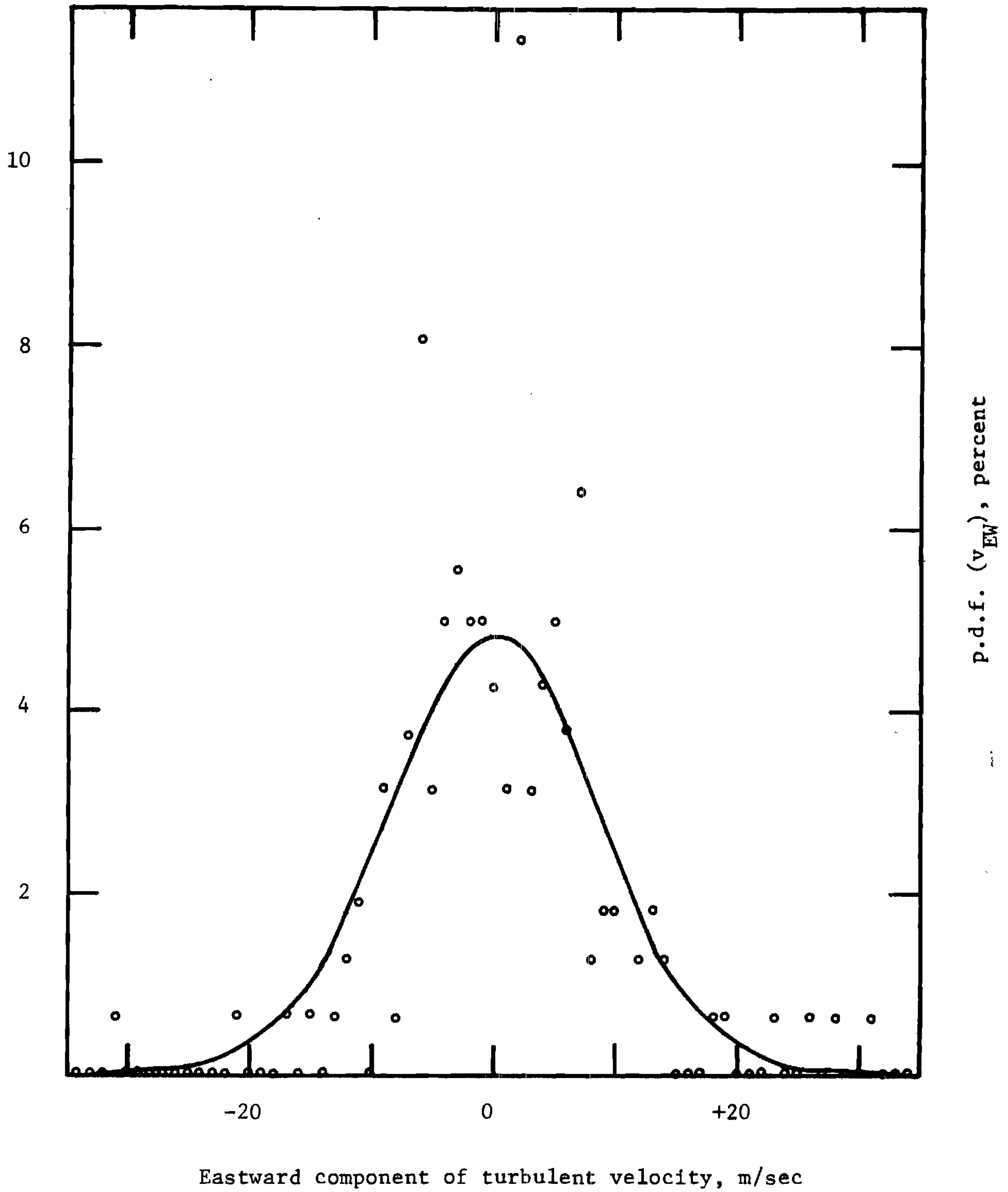

Figure 7. The p.d.f. for the Eastward Component of the Turbulent Velocity in the 90-95 km Region. (The smooth curve is a Gaussian distribution with the same standard deviation and the same mean as the observed distribution.)

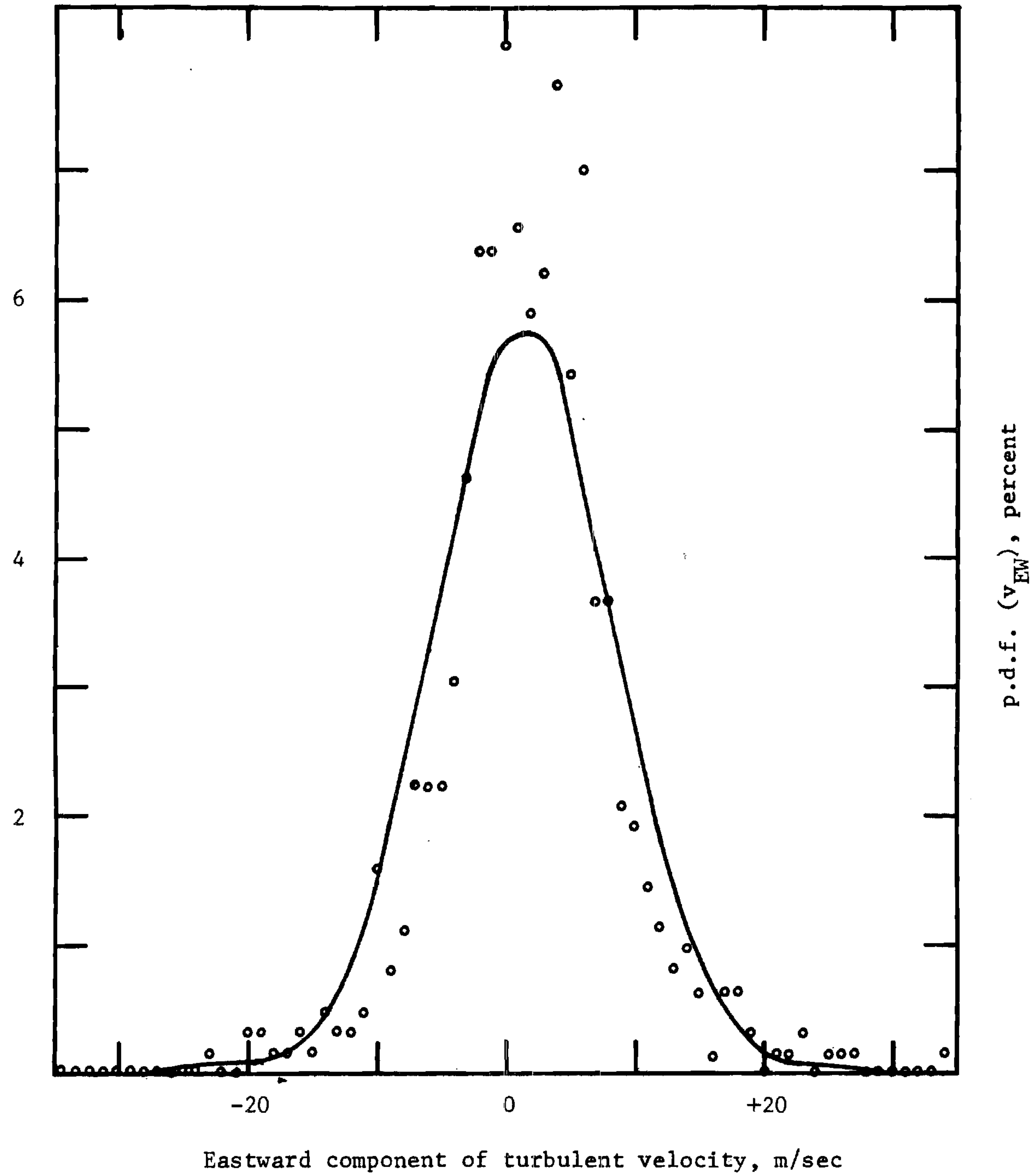

Figure 8. The p.d.f. for the Eastward Component of the Turbulent Velocity in the 95-100 km Region. (The smooth curve is a Gaussian distribution with the same standard deviation and the same mean as the observed distribution.)

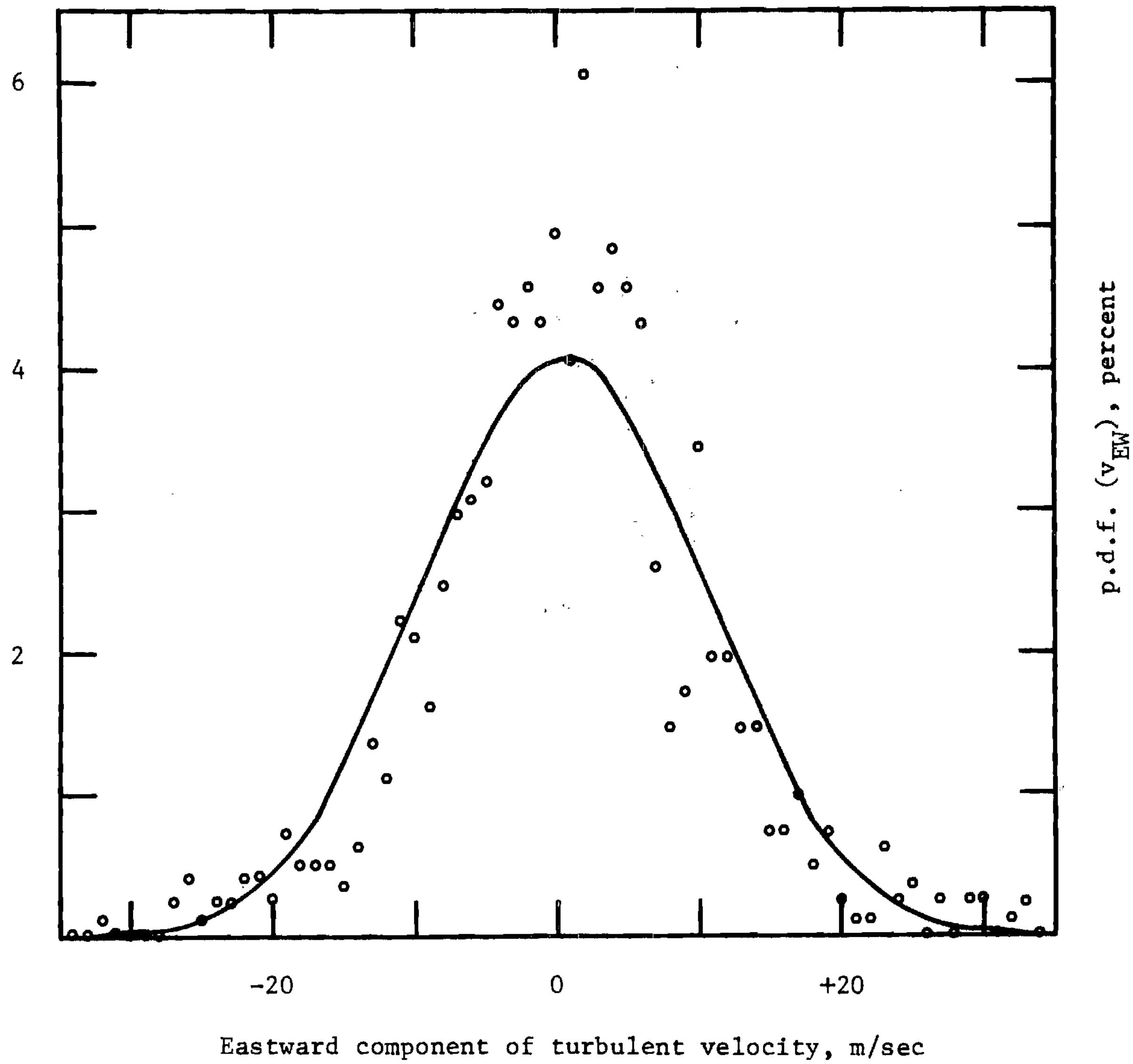

Figure 9. The p.d.f. for the Eastward Component of the Turbulent Velocity in the 100-105 km Region. (The smooth curve is a Gaussian distribution with the same standard deviation and the same mean as the observed distribution.)

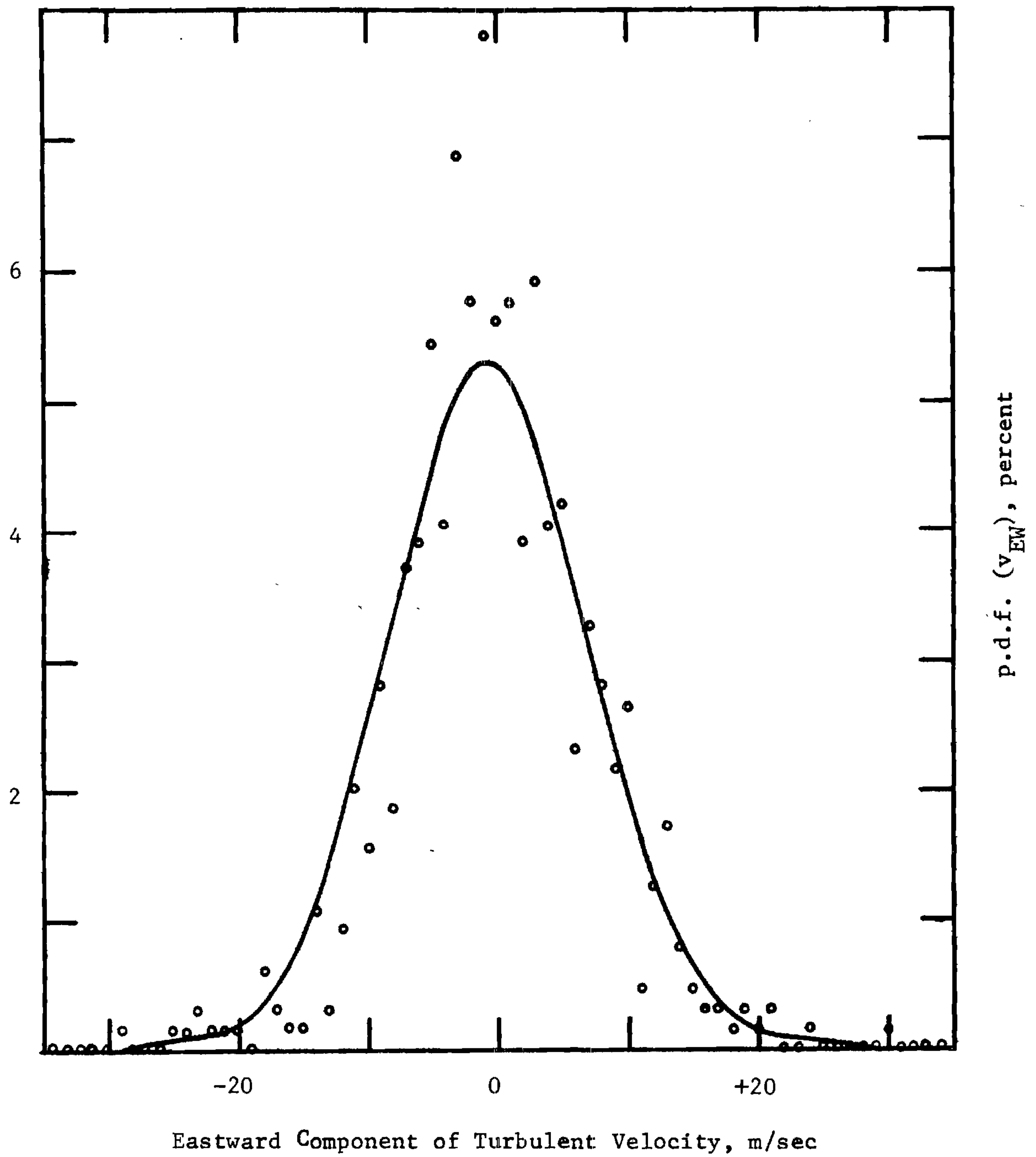

Figure 10. The p.d.f. for the Eastward Component of the Turbulent Velocity in the 105-110 km Region. (The smooth curve is a Gaussian distribution with the same standard deviation and the same mean as the observed distribution.)

and vertical components of the turbulent velocity in each of the four altitude intervals were extremely similar, they were omitted. There were 161 observations in the 90-95 km range, 627 in the 95-100 km range, 808 in the 100-105 km range, and 641 in the 105-110 km range. Table 2 gives the calculated values of β_2 for each of the components in the four altitude intervals. A t-test using the statistic given in equation 2.3 was applied to each member of the table. As can be seen, every member of the table was very significantly different from the Gaussian value 3.0. Thus the non-Gaussian characteristic of a high β_2 is maintained throughout each altitude interval by all components. Table 3 gives the values for $\langle v \rangle$ and σ for each of the components in the four altitude intervals.

Table 4 shows the calculated values of γ_1 for each of the turbulent velocity components in the four altitude intervals. A t-test using the statistic given in equation 2.2 was performed on each value in the table. No velocity component p.d.f. is consistently skewed at all altitudes. However, in three of the four altitude intervals, at least one of the velocity components had a highly significantly skewed p.d.f. The data given above have also been discussed in a paper by Moseley and Justus [1967].

Due to the inconsistent character of the skewness, it was felt that a coordinate system which was more relevant to the flow field should be used. On the basis of theoretical work to be discussed in Chapter 3 and because of research by Justus [1968] on turbulence symmetric about a plane, the new coordinate system which was chosen was the shear system. Shear is defined to be the distance rate of

Table 2. Values of β_2 in Geographic System. (An asterisk indicates significance at 1% level.)

Altitude Interval in km.	Northward Component of Turbulent Velocity	Eastward Component of Turbulent Velocity	Vertical Component of Turbulent Velocity
90- 95	6.952*	5.426*	4.488*
95-100	3.802*	5.289*	4.499*
100-105	4.525*	4.473*	4.457*
105-110	5.439*	5.099*	4.228*

Table 3. Values of <v> and σ in the Geographic System. (An asterisk indicates significance at 1% level, and □ indicates significance at 5% level.)

Altitude Interval in km.	Northward Component of Turbulent Velocity		Eastward Component of Turbulent Velocity		Vertical Component of Turbulent Velocity	
	<v>,m/sec	σ,m/sec	<v>,m/sec	σ,m/sec	<v>,m/sec	σ,m/sec
90- 95	1.764*	7.693	-0.009	8.351	0.621	8.662
95-100	0.698	7.791	1.365*	6.903	1.190*	7.131
100-105	-0.286	7.850	0.417	9.786	3.726*	8.836
105-110	0.366	10.220	-0.708□	7.557	2.142*	9.193

Table 4. Values of γ_1 in the Geographic System. (An asterisk indicates significance at 1% level, and □ indicates significance at 5% level.)

Altitude Interval in km.	Northward Component of Turbulent Velocity	Eastward Component of Turbulent Velocity	Vertical Component of Turbulent Velocity
90- 95	0.742*	0.286	-0.486□
95-100	0.080	0.280*	0.244□
100-105	0.014	0.278*	0.108
105-110	-0.026	0.182	-0.108

change of the wind velocity. In the most general situation, the shear would be a second order tensor with elements denoted by $\frac{\partial V_i}{\partial x_j}$ (i = 1,2,3 and j = 1,2,3). In the atmospheric case, the vertical mean velocity is considered to be zero, thereby eliminating three of the nine elements. The horizontal shear components (the derivatives of the horizontal mean winds with respect to the horizontal directions) are small compared to the vertical shear components (the derivatives of the horizontal mean winds with respect to the vertical direction). The horizontal shear components were determined by Rosenberg and Justus [1965] to be about 0.05 m/sec/km, whereas the vertical shear components have an average magnitude of about 20.5 m/sec/km. Thus the horizontal shear components are neglected leaving only the two vertical shear components, $\frac{\partial V_N}{\partial z}$ and $\frac{\partial V_E}{\partial z}$.

Data in the Shear System

The northward and eastward components of the mean wind velocity profile, $V_N(z)$ and $V_E(z)$, have already been determined. From these one can determine the northward and eastward components of the mean shear profile, i.e. $\frac{\partial V_N}{\partial z}$ and $\frac{\partial V_E}{\partial z}$. This determination is made in the following manner. In order to find the value of $\frac{\partial V_N}{\partial z}$ at a particular altitude, say z_0, a fourth order polynomial is fitted to five values of V_N at five altitudes whose interval contains the altitude z_0. Then by differentiating the resulting polynomial for V_N with respect to z and evaluating at z_0, one arrives at the value of $\frac{\partial V_N}{\partial z}$ at z_0. This procedure is continued until a profile of $\frac{\partial V_N}{\partial z}$ versus altitude is obtained. Then an identical procedure using $V_E(z)$ gives the profile

$\frac{\partial V_E}{\partial z}$ versus altitude.

At a given altitude, one can treat the two remaining components of the shear tensor as components of a shear vector lying in a horizontal plane at that altitude. Thus the shear vector $\frac{\partial V}{\partial z}$ would have the two components $\frac{\partial V_N}{\partial z}$ and $\frac{\partial V_E}{\partial z}$ along the northward and eastward axes respectively. Viewed as a vector, the shear would have a magnitude and a direction. The square of the magnitude of the shear vector is given by

$$\left(\frac{\partial V}{\partial z}\right)^2 = \left(\frac{\partial V_N}{\partial z}\right)^2 + \left(\frac{\partial V_E}{\partial z}\right)^2 \tag{2.4}$$

and the direction is given by the angle ϕ where

$$\phi = \cos^{-1}\left(\frac{\partial V_E}{\partial z}\Big/\frac{\partial V}{\partial z}\right) \tag{2.5}$$

It should be noted that both $\frac{\partial V}{\partial z}$ and ϕ are functions of altitude, also ϕ is not the heading of the shear vector since the heading is the number of degrees east of north.

At each altitude, the shear system is defined so that the x_1 axis is along the direction of the shear vector at that altitude, the x_2 axis is perpendicular to the direction of the shear vector and still in the horizontal plane, and the x_3 axis is in the vertical direction. Figure 11 illustrates the alignment of the shear system. Since the angle ϕ is a function of altitude, then the orientation of the shear system is also a function of altitude. It must also be

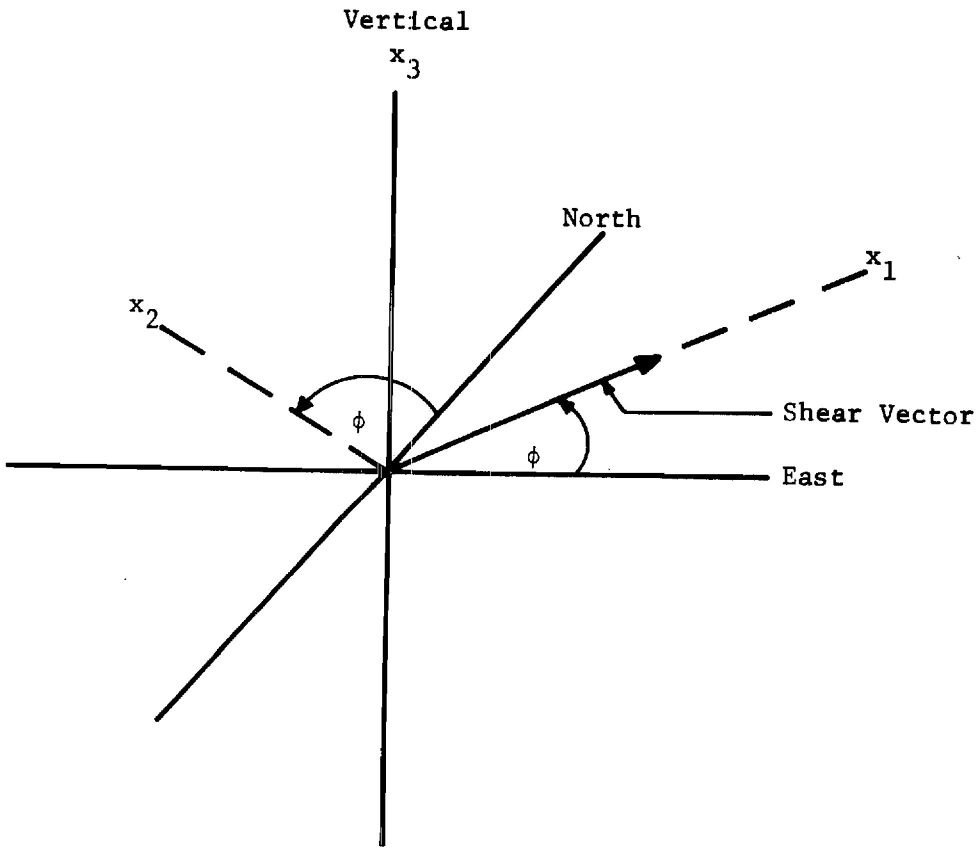

Figure 11. The Shear System

recognized that since the mean wind profiles vary from one rocket probe to the next rocket probe (these rocket probes were launched throughout several years), the orientation of the shear system at a particular altitude will vary from rocket to rocket.

Now if the particular rocket probe is given (which establishes the proper profiles of $\frac{\partial V_N}{\partial z}$ and $\frac{\partial V_E}{\partial z}$ to use) and if the altitude is known (from which the proper values of $\frac{\partial V_N}{\partial z}$ and $\frac{\partial V_E}{\partial z}$ can be determined), then the turbulent velocity components along the axes of the shear system will be given by the sum of the projections of the northward and eastward turbulent velocity components upon the shear system axes. Thus

$$v_1 = v_{EW} \cos \phi + v_{NS} \sin \phi \tag{2.6}$$

$$v_2 = v_{NS} \cos \phi - v_{EW} \sin \phi \tag{2.7}$$

$$v_3 = v_{Vert} \tag{2.8}$$

where v_1 is the turbulent velocity component along the x_1 axis of the shear system, v_2 is the turbulent velocity component along the x_2 axis of the shear system, and v_3 is the turbulent velocity component along the x_3 axis of the shear system.

Since the instantaneous vertical component of the turbulent velocity can be non-zero, the altitude of a distinguishable cloud feature can vary randomly. This variation is not large. The altitude to be used in determining the orientation of the shear system for a particular globule is taken to be the average altitude of the globule over the duration of the cloud.

After all of the turbulent velocity observations were properly converted to the shear system, the p.d.f.'s for the three components were computed. Figure 12 shows the p.d.f. of the perpendicular component (v_2) of the velocity (designated perpendicular component because the x_2 axis is perpendicular to the direction of the shear). The smooth curve in Figure 12 is again a Gaussian p.d.f. with the same mean and standard deviation as the experimental distribution. Figures for the parallel component (v_1) of the turbulent velocity and the vertical component (v_3) of the turbulent velocity were omitted because they were very similar to Figure 12. As is shown in the figure, the p.d.f. for v_2 has a high central peak and broad tails. Thus it will have a large β_2 and will be non-Gaussian. The number of observations represented in Figure 12 and used in the computations for Table 5 is 2245. The reason for the difference in this number and the number of observations in the geographic system (2308) is that conversion of observations to the shear system was undefined when the magnitude of the shear vector was zero.

Following computation of the p.d.f.'s in the shear system, various moments of the p.d.f.'s were computed. Table 5 gives the calculated values for σ, $\langle v \rangle$, γ_1 and β_2 for the three components. t-tests were also performed on each of the values for γ_1 and β_2 (where the statistics for the t-tests are given by equations 2.2 and 2.3 respectively) and their results are shown in Table 5. The standard definitions for high significance and significance were applied. It

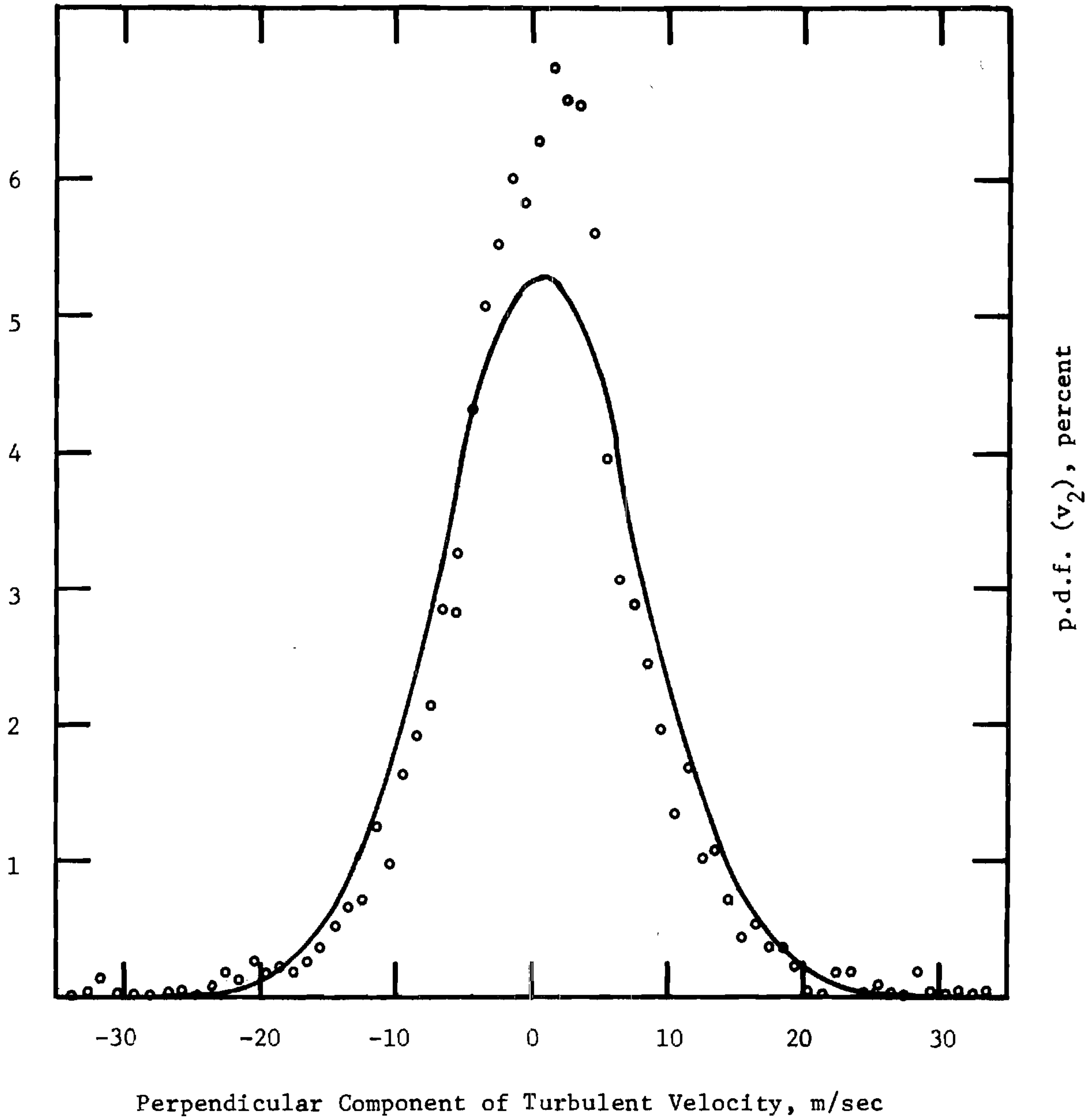

Figure 12. The p.d.f. for the Perpendicular Component of the Turbulent Velocity for all Data in the Shear System. (The smooth curve is a Gaussian distribution with the same standard deviation and the same mean as the observed distribution.)

Table 5. Parameters of the Observed p.d.f.'s for the Three Turbulent Velocity Components in the Shear System. (Asterisks (*) indicate values that are very significantly different from the expected value for a Gaussian distribution. 'Highly significant difference' means better than 1% significance level based on a t-test.)

Parameter	~~Expected Gaussian~~ Value	Parallel Velocity Component (v_1)	Perpendicular Velocity Component (v_2)	Vertical Velocity Component (V_3)
σ, m/sec	σ	9.295	7.496	8.502
<v>,m/sec	0	-0.850	0.654	2.196
γ_1	0	0.068	-0.094	0.022
β_2	3	4.968*	5.624*	4.655*

is immediately obvious from Table 5 that (in the shear system for the combined observations from all altitudes and magnitudes of the shear) none of the p.d.f.'s for the three components is significantly skewed and that the non-Gaussian character of the p.d.f.'s is consistently maintained for all of the three components by highly significant values of β_2.

As in the geographic system, the observations in the shear system were divided into four altitude intervals of five km each and the p.d.f.'s for each component of the velocity were computed. The altitude intervals were 90-95 km, 95-100 km, 100-105 km, and 105-110 km. Figure 13, 14, 15, and 16 show the p.d.f.'s for the parallel component of the velocity. The smooth curve on each figure once again represents a Gaussian p.d.f. with a mean and standard deviation appropriate to each figure's data. Corresponding figures for the p.d.f.'s of the perpendicular and vertical components of the velocity were omitted since they were very similar. There were 159 observations in the 90-95 km range, 618 in the 95-100 km range, 793 in the 100-105 km range, and 623 in the 105-110 km range. Several moments were computed for each of the p.d.f.'s and t-tests were applied to the values for γ_1 and β_2. Table 6 gives the values of β_2 for each component in each altitude interval for the shear system. The values of β_2 are significantly different from the Gaussian value for all cases. Table 7 is a list of the values of $\langle v \rangle$ and σ for each of the components in each of the altitude intervals. Table 8 provides the values of the skewness γ_1 for each of the components and gives the average value of the shear magnitude in each of the altitude intervals. All three

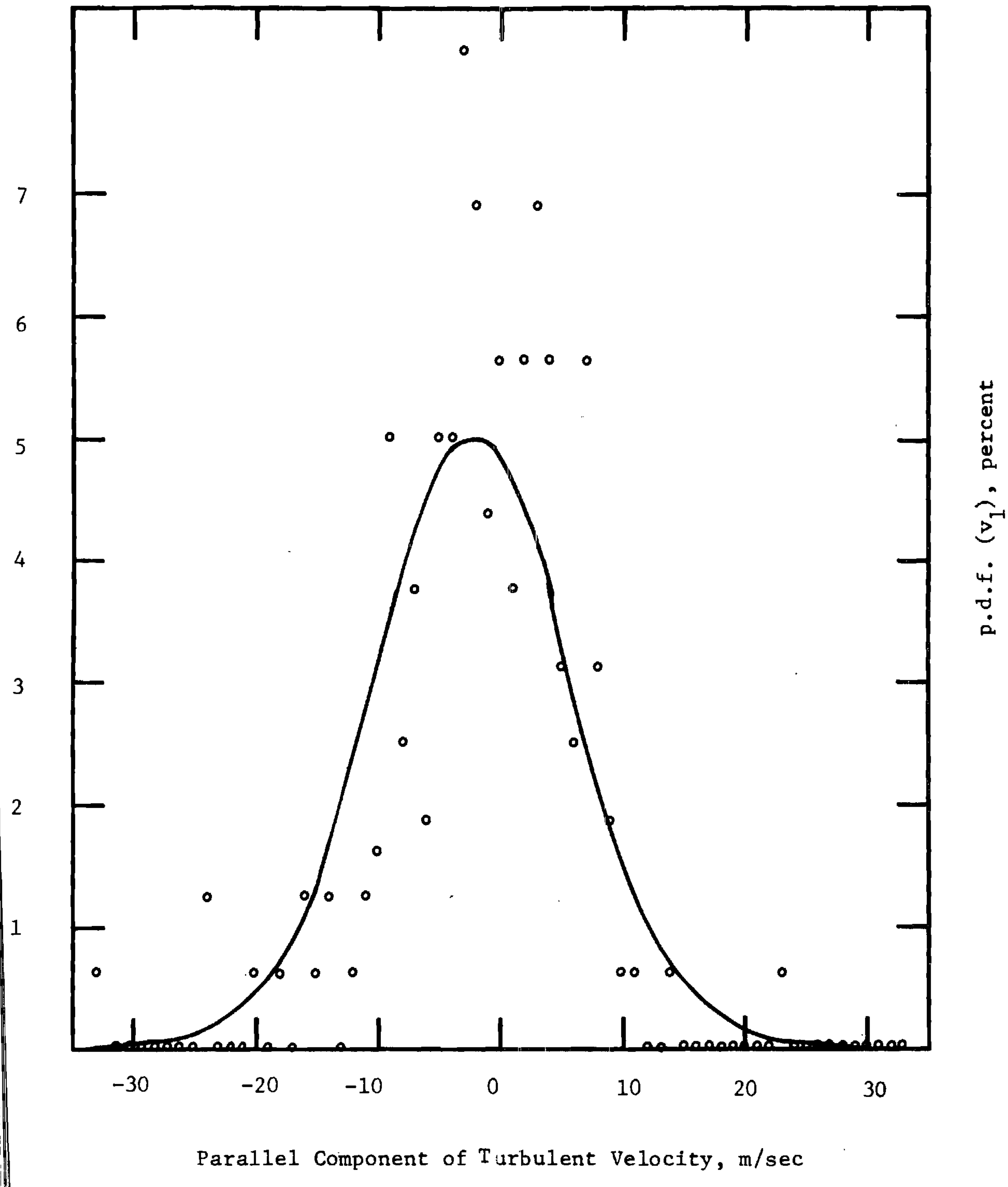

Figure 13. The p.d.f. for the Parallel Component of the Turbulent Velocity in the 90-95 km Altitude Region. (The smooth curve is a Gaussian distribution with the same standard deviation and the same mean as the observed distribution.)

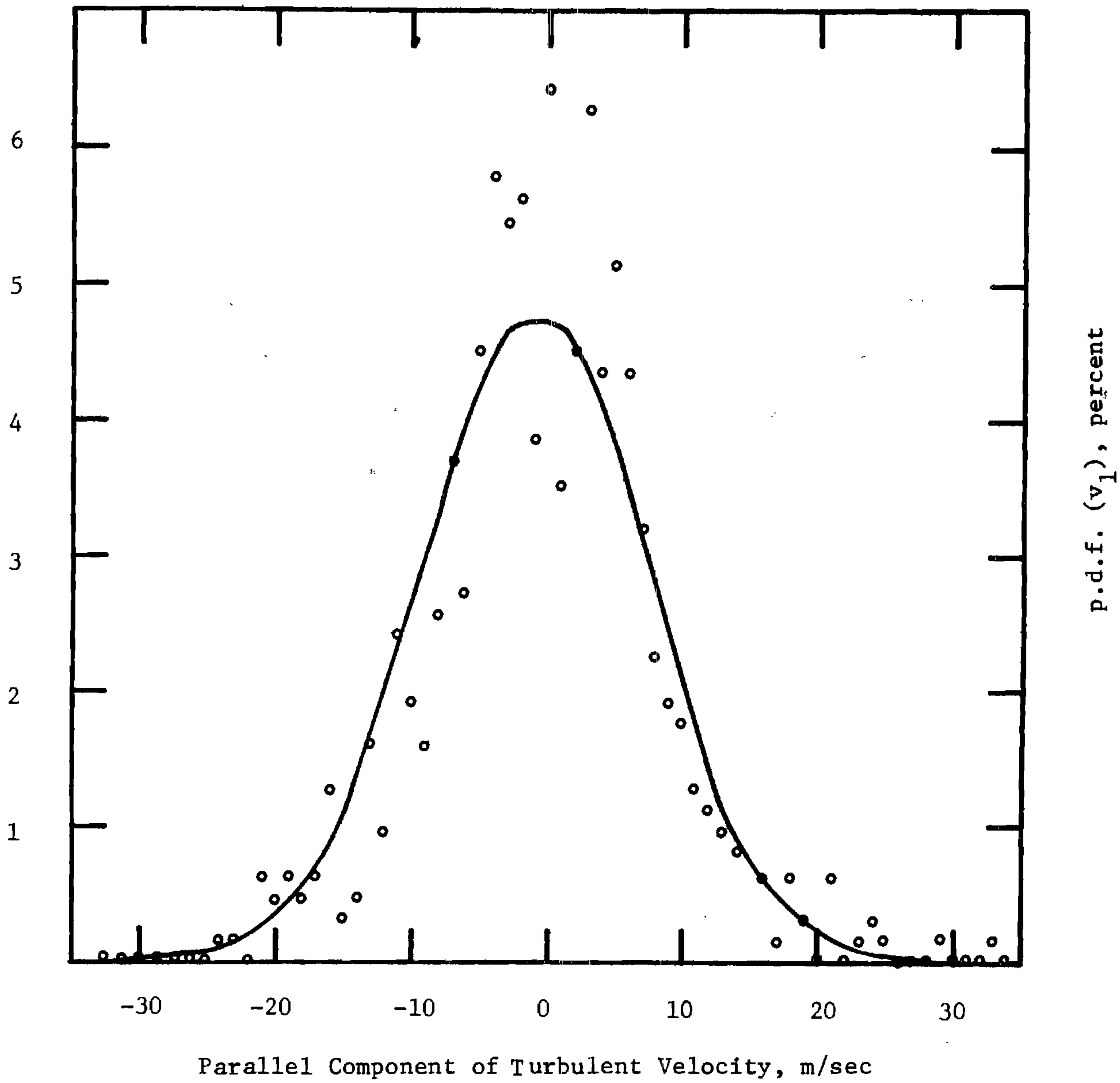

Figure 14. The p.d.f. for the Parallel Component of the Turbulent Velocity in the 95-100 km Altitude Region. (The smooth curve is a Gaussian distribution with the same standard deviation and the same mean as the observed distribution.)

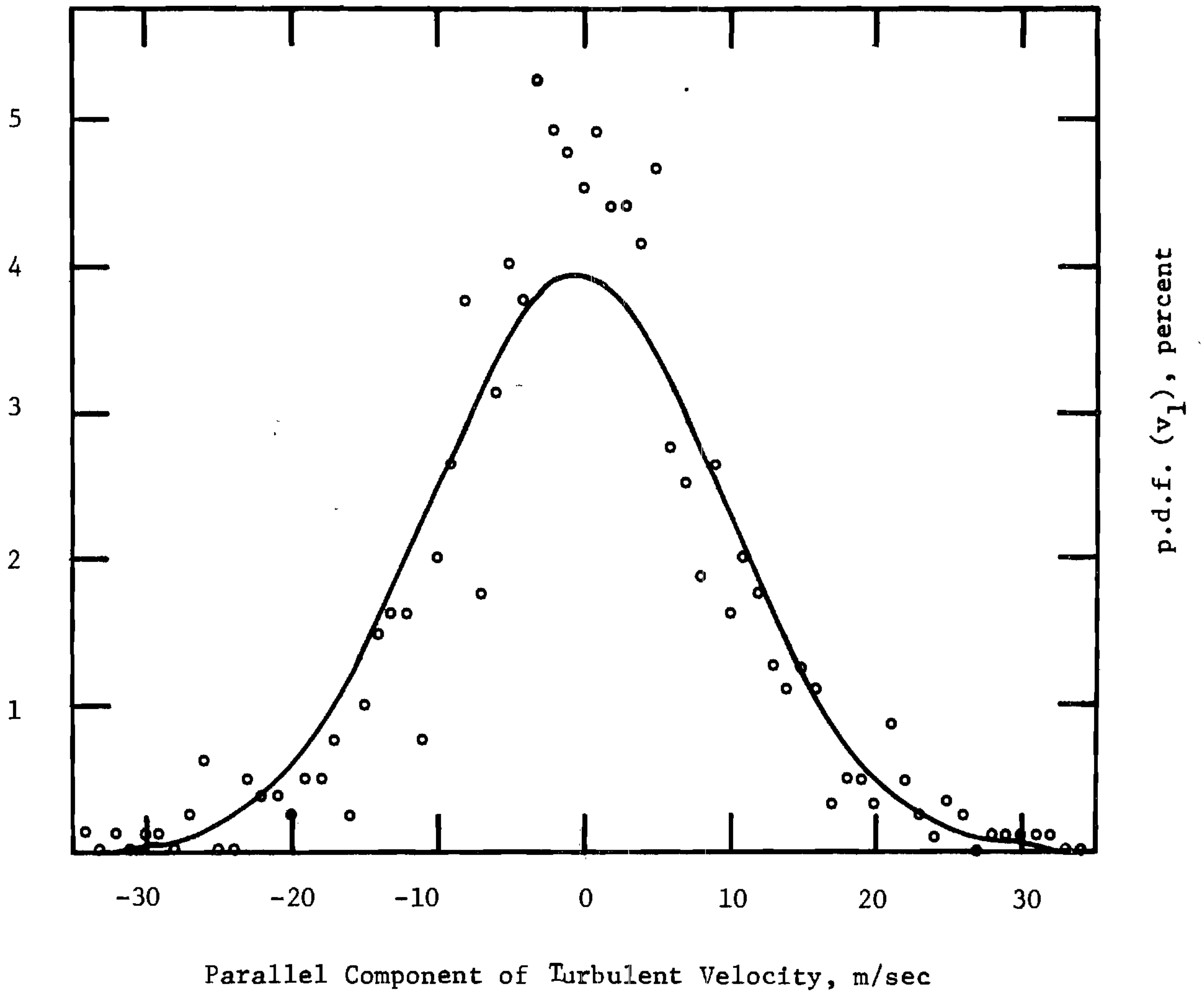

Figure 15. The p.d.f. for the Parallel Component of the Turbulent Velocity in the 100-105 km Altitude Region. (The smooth curve is a Gaussian distribution with the same standard deviation and the same mean as the observed distribution.)

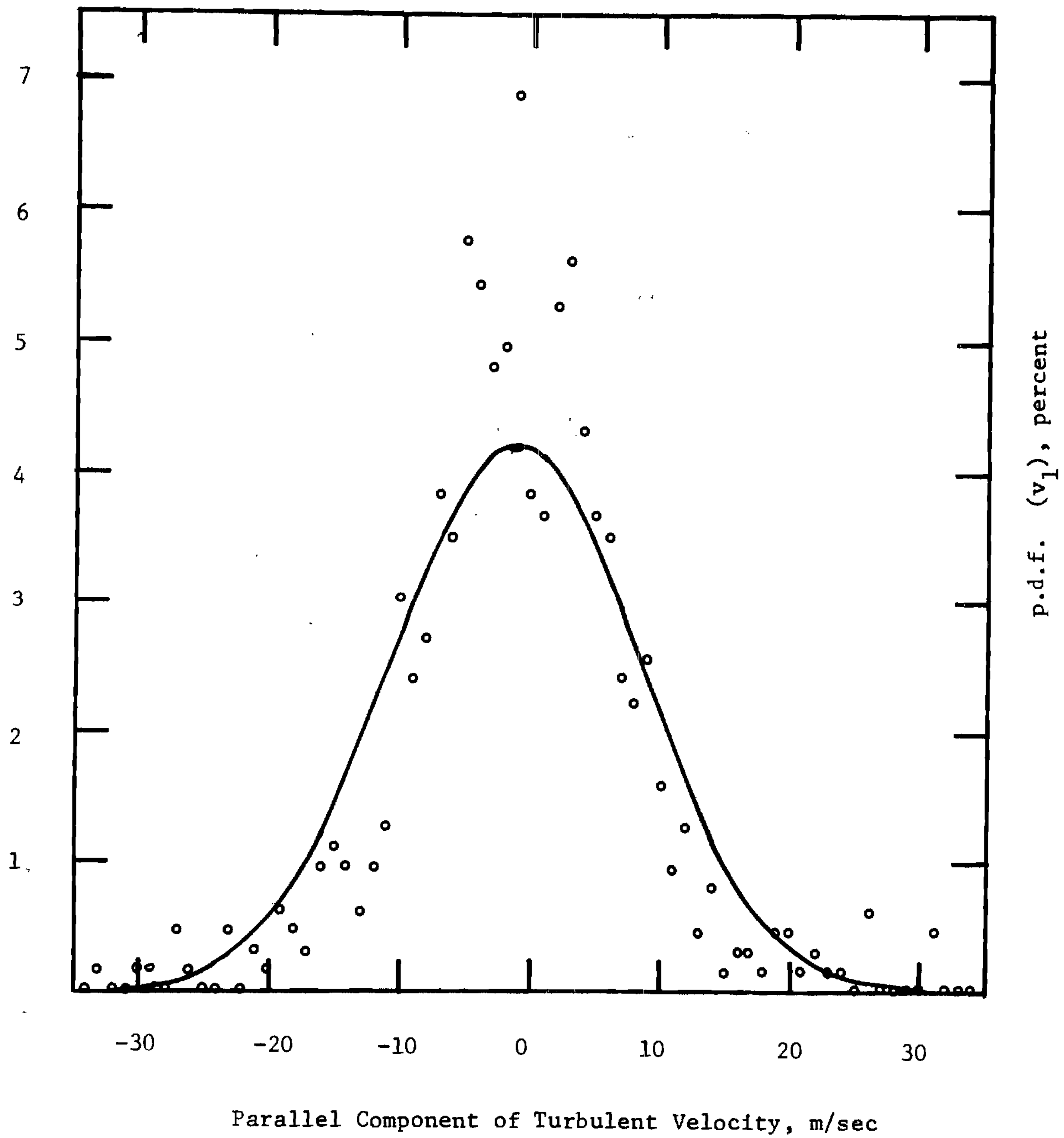

Figure 16. The p.d.f. for the Parallel Component of the Turbulent Velocity in the 105–110 km Altitude Region. (The smooth curve is a Gaussian distribution with the same standard deviation and the same mean as the observed distribution.)

Table 6. Values of β_2 in the Shear System for Altitude Intervals.

(An Asterisk indicates significance at 1% level.)

Altitude Interval in km.	Average Shear Magnitude in m/sec/km.	Parallel Component of Velocity (v_1) β_2	Perpendicular Component of Velocity (v_2) β_2	Velocity Component of Velocity (v_3) β_2
90- 95	16.58	6.407*	7.091*	4.580*
95-100	20.47	3.896*	5.407*	4.557*
100-105	23.26	4.210*	5.039*	4.593*
105-110	24.18	6.089*	5.667*	4.372*

Table 7. Values of <v> and σ in the Shear System for Altitude Intervals.

Altitude Interval in km.	Average Shear Magnitude in m/sec/km.	Parallel Component of velocity (v_1) <v>,m/sec	Parallel Component of velocity (v_1) σ,m/sec	Perpendicular Component of velocity (v_2) <v>,m/sec	Perpendicular Component of velocity (v_2) σ,m/sec	Vertical Component of velocity (v_3) <v>,m/sec	Vertical Component of velocity (v_3) σ,m/sec
90- 95	16.58	-2.169	7.879	1.142	7.678	0.355	8.382
95-100	20.47	-0.806	8.215	1.627	6.214	1.148	7.067
100-105	23.26	-0.448	10.050	0.027	7.438	3.614	8.737
105-110	24.18	-1.148	9.444	0.529	8.362	1.860	8.936

Table 8. Values of γ_1 in the Shear System for Altitude Intervals. (An asterisk indicates significance at 1% level, and □ indicates significance at 5% level.)

Altitude Interval in km.	Average Shear Magnitude in m/sec/km.	Parallel Component of velocity (v_1) γ_1	Perpendicular Component of velocity (v_2) γ_1	Vertical Component of velocity (v_3) γ_1
90- 95	16.58	-1.057*	0.434□	-0.656*
95-100	20.47	0.1349	-0.1079	0.2307□
100-105	23.26	-0.1027	0.1158	0.0854
105-110	24.18	0.3069*	-0.2558*	0.1248

components were significantly skewed in the 90-95 km interval, but only the vertical component had a large value of the skewness in the 95-100 km interval. No component is skewed in the 100-105 km range, but the parallel and the perpendicular components are skewed in the 105-110 km range. Thus some degree of inconsistency still remains in the altitude breakup.

Next, the observations in the shear system were divided into 5 shear magnitude intervals. The intervals were chosen such that each contained approximately 425 observations. The intervals used were the following: 0-12 m/sec/km, 12-17 m/sec/km, 17-21 m/sec/km, 21-29 m/sec/km, 29-50 m/sec/km. The numbers of observations in these intervals were 393, 456, 442, 451 and 409 respectively. The average values of the magnitude of the shear vector in the intervals were 8.63 m/sec/km, 14.36 m/sec/km, 19.11 m/sec/km, 24.36 m/sec/km, and 35.96 m/sec/km. Once the observations had been placed in the proper shear magnitude interval, the p.d.f.'s for each component in each shear magnitude interval were computed; and several moments were calculated for each p.d.f. Then t-tests were evaluated for each of the values of γ_1 and β_2. Figures 17, 18, 19, 20 and 21 show the p.d.f.'s for the perpendicular component of the turbulent velocity. The smooth curve represents a Gaussian p.d.f. with the mean and standard deviation appropriate to the observations in the particular shear magnitude interval. Figures for the p.d.f.'s of the parallel and vertical components were omitted because of their similarity to the perpendicular component p.d.f.'s.

Table 9 gives the values of β_2 for the three components in the

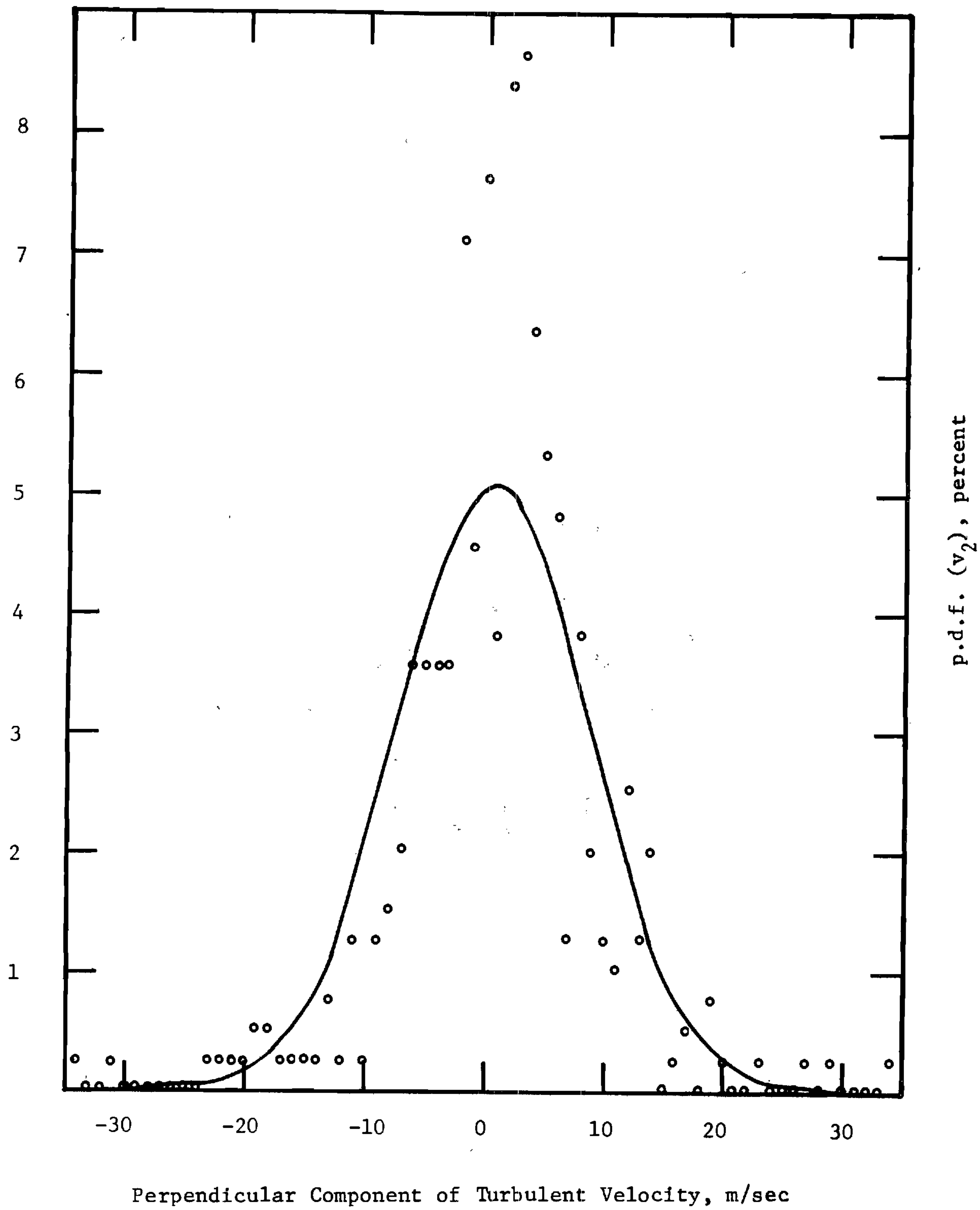

Figure 17. The p.d.f. for the Perpendicular Component of the Turbulent Velocity in the 0-12 m/sec/km Shear Region. (The smooth curve is a Gaussian distribution with the same standard deviation and the same mean as the observed distribution.)

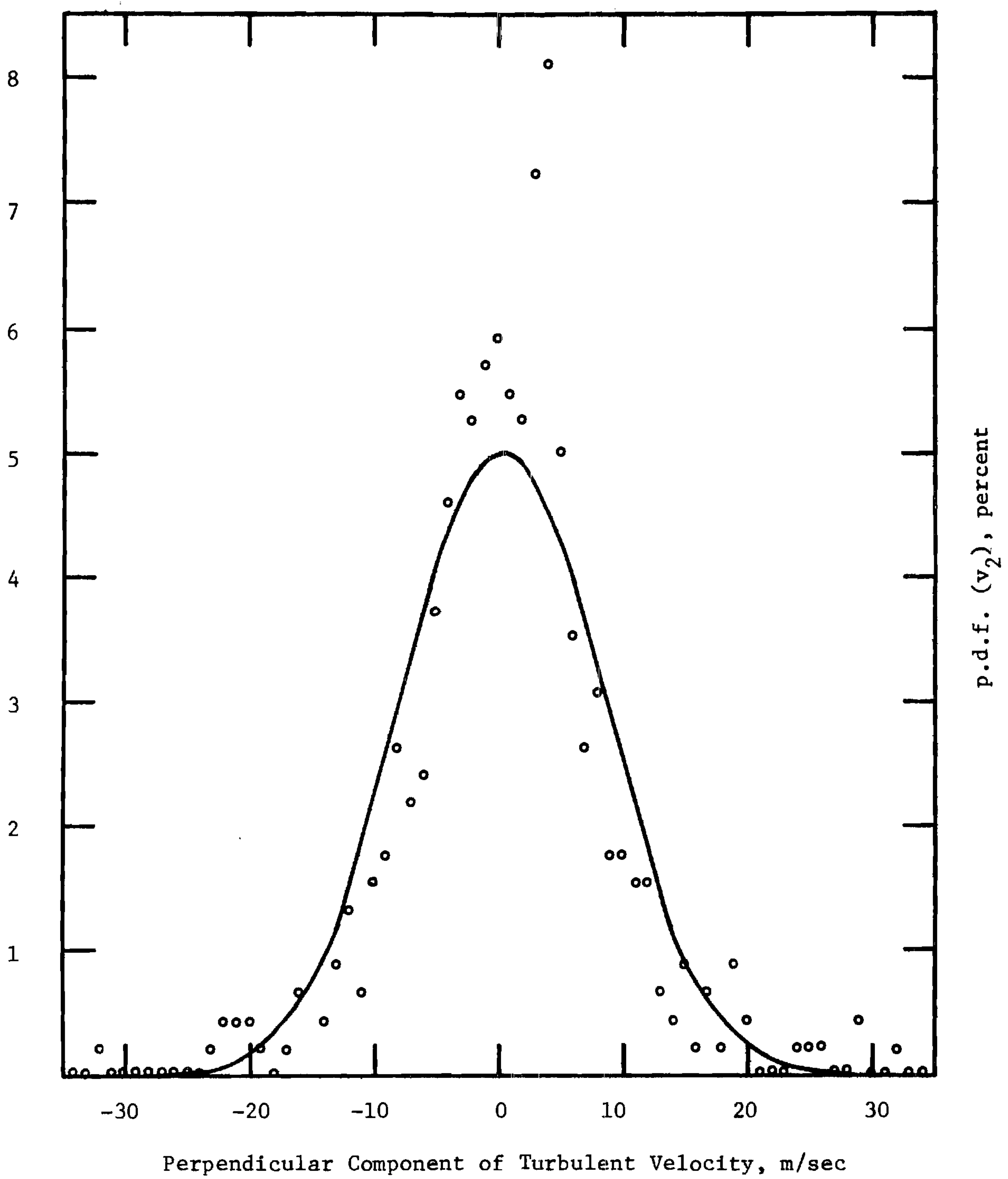

Figure 18. The p.d.f. for the Perpendicular Component of the Turbulent Velocity in the 12-17 m/sec/km Shear Region. (The smooth curve is a Gaussian distribution with the same standard deviation and the same mean as the observed distribution.)

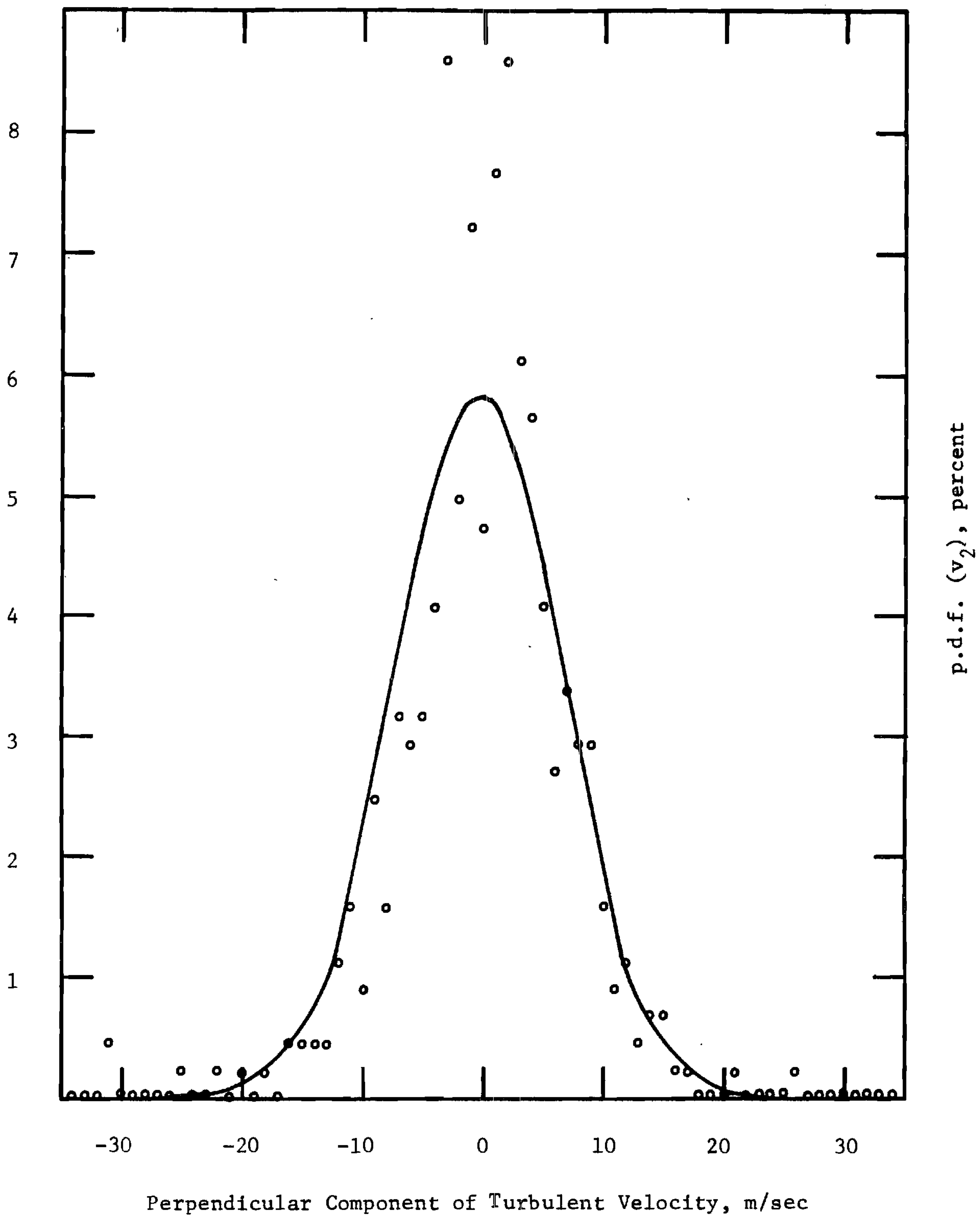

Figure 19. The p.d.f. for the Perpendicular Component of the Turbulent Velocity in the 17-21 m/sec/km Shear Region. (The smooth curve is a Gaussian distribution with the same standard deviation and the same mean as the observed distribution.)

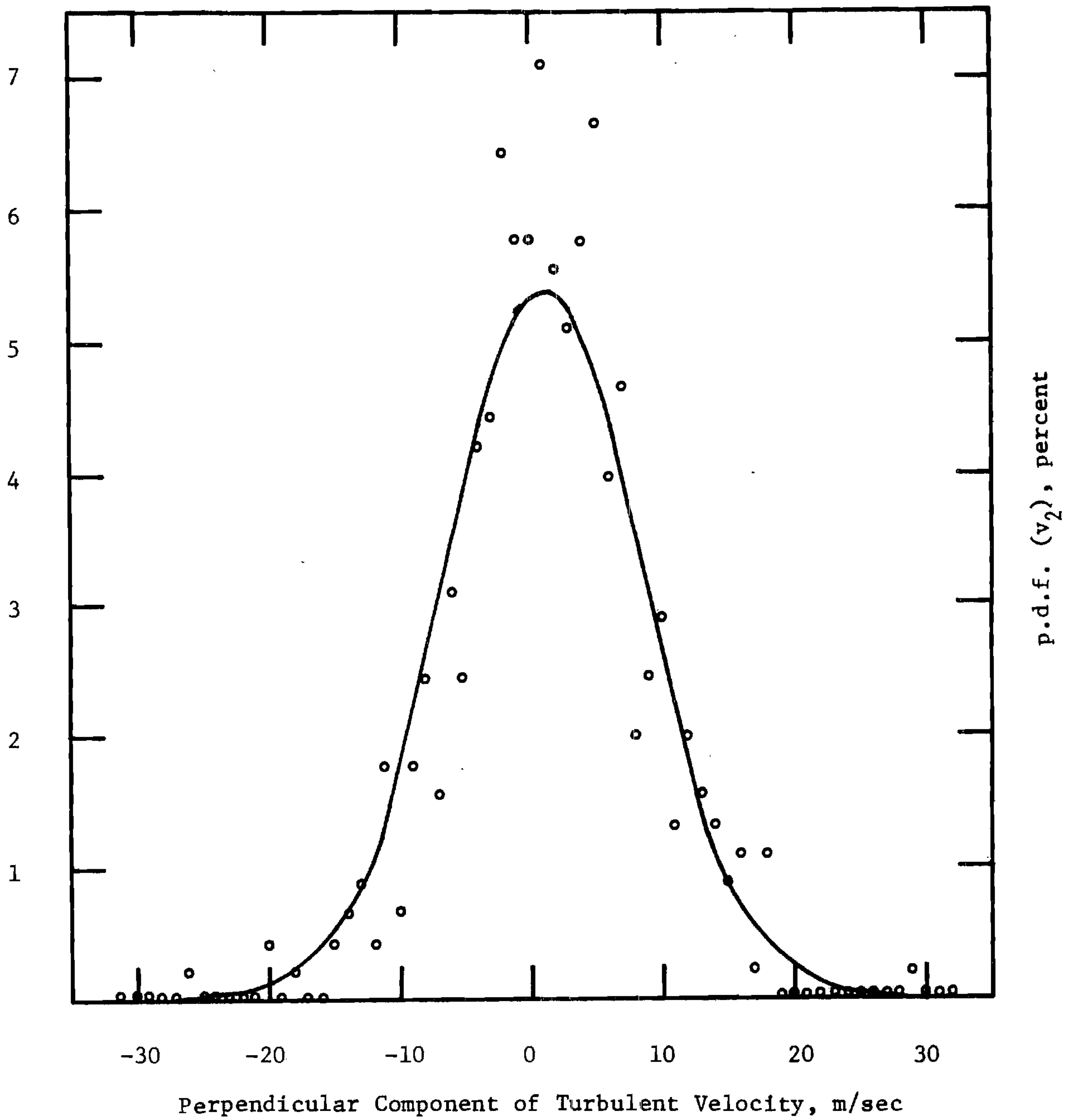

Figure 20. The p.d.f. for the Perpendicular Component of the Turbulent Velocity in the 21-29 m/sec/km Shear Region. (The smooth curve is a Gaussian distribution with the same standard deviation and the same mean as the observed distribution.)

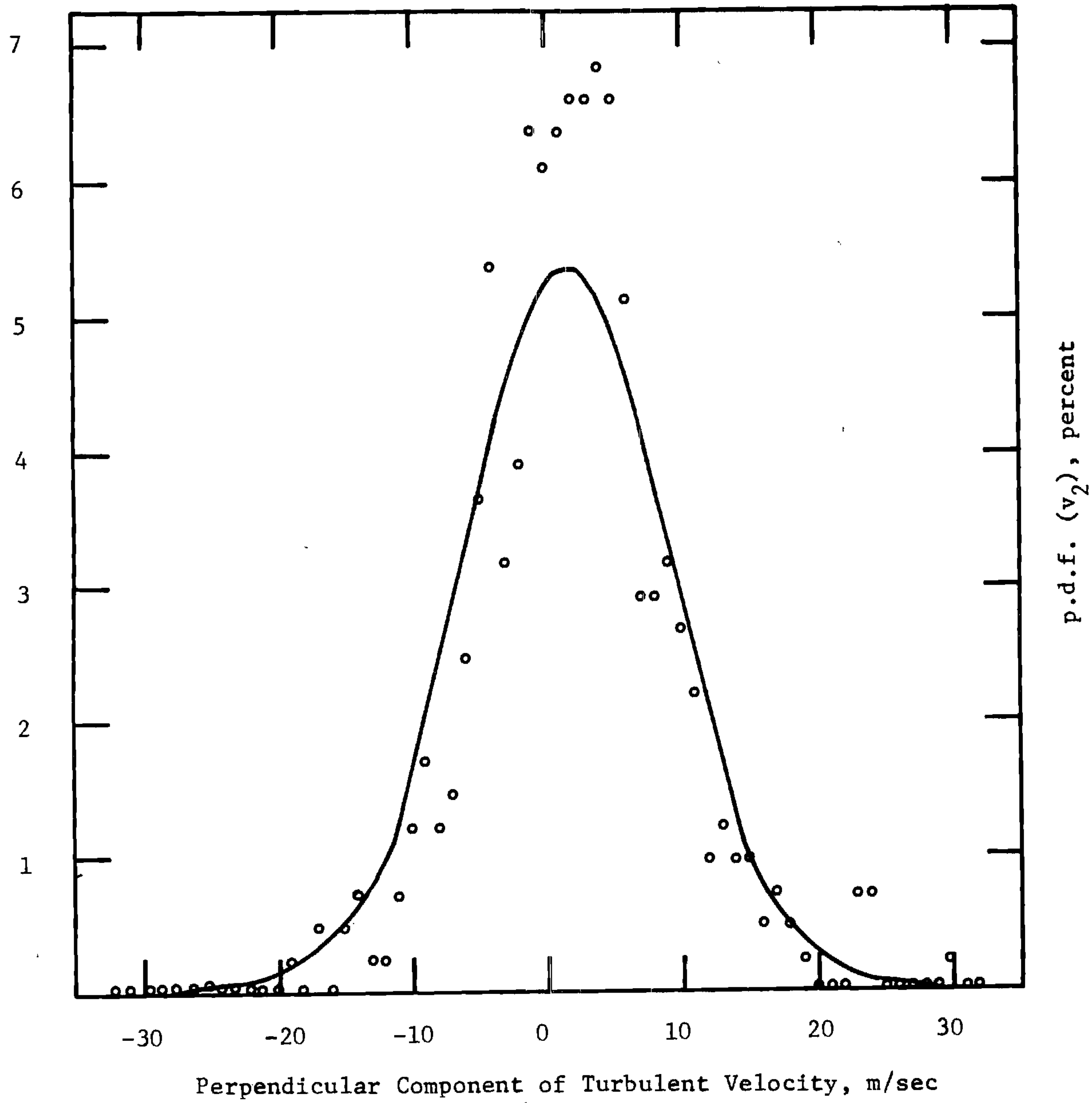

Figure 21. The p.d.f. for the Perpendicular Component of the Turbulent Velocity in the 29-50 m/sec/km Shear Region. (The smooth curve is a Gaussian distribution with the same standard deviation and the same mean as the observed distribution.)

Table 9. Values of β_2 in the Shear System for Shear Magnitude Intervals. (An asterisk indicates significance at 1% level, and □ indicates significance at 5% level.)

Shear Magnitude Interval in m/sec/km	Average Shear Magnitude in m/sec/km.	Parallel Component of velocity (v_1) β_2	Perpendicular Component of velocity (v_2) β_2	Vertical Component of velocity (v_3) β_2
0 - 12	8.63	4.849*	5.892*	4.509*
12 - 17	14.36	4.993*	5.011*	4.560*
17 - 21	19.11	5.169*	5.449*	5.543*
21 - 29	24.36	5.058*	4.929*	3.297*
29 - 50	35.96	3.570□	7.431*	4.540*

shear magnitude intervals. It indicates that in every case the value of β_2 was significantly higher than the Gaussian value. Thus the non-Gaussian character of the p.d.f.'s is consistently maintained.

Table 10 provides a list of the mean velocity $\langle v \rangle$ and σ for all three components in each shear magnitude interval. Table 11 gives the values of γ_1 for the three components in all of the shear magnitude intervals. As is indicated, none of the p.d.f.'s for the components is significantly skewed in the first two shear magnitude intervals where the shear magnitude is below 17 m/sec/km. However, in each of the three intervals where the shear magnitude was larger than 17 m/sec/km, the p.d.f.'s for two of the components are significantly skewed. The p.d.f. for the perpendicular component is significantly skewed in all three of these latter intervals. The p.d.f. for the vertical component is significantly skewed in two of the three latter intervals, namely the 17-21 m/sec/km interval and the 29-50 m/sec/km interval. The 21-29 m/sec/km has a significantly skewed p.d.f. for the parallel component. Thus in the shear system with a division of the observations into shear magnitude intervals, some consistency in the skewness of the p.d.f.'s is obtained, and the non-Gaussian character of the high flatness factor is completely maintained.

Statistical Tests

Another statistical test was performed to further substantiate the non-Gaussian character of the data. The chi-square goodness-of-fit test was the test employed. As is noted in Appendix A, the χ^2-test is a non-parametric test and thus the validity of the test would not be

Table 10. Values of <v> and σ in the Shear System for Shear Magnitude Intervals.

Shear Magnitude Interval in m/sec/km	Average Shear Magnitude in m/sec/km	Parallel Component of velocity (v_1)		Perpendicular Component of velocity (v_2)		Vertical Component of velocity (v_3)	
		<v>,m/sec	σ,m/sec	<v>,m/sec	σ,m/sec	<v>,m/sec	σ,m/sec
0 - 12	8.63	-1.320	8.522	0.706	7.819	2.784	9.569
12 - 17	14.36	-1.348	8.295	0.359	7.919	3.318	9.276
17 - 21	19.11	-0.451	8.504	-0.309	6.810	2.564	7.599
21 - 29	24.36	-1.298	9.128	1.139	7.361	1.757	7.727
29 - 50	35.96	-0.129	9.892	1.560	7.435	0.941	7.743

Table 11. Values of γ_1 in the Shear System for Shear Magnitude Intervals. (An asterisk indicates significance at 1% level, and □ indicates significance at 5% level.)

Shear Magnitude Interval in m/sec/km.	Average Shear Magnitude in m/sec/km	Parallel Component of velocity (v_1) γ_1	Perpendicular Component of velocity (v_2) γ_1	Vertical Component of velocity (v_3) γ_1
0 - 12	8.63	0.1638	-0.252	-0.0378
12 - 17	14.36	-0.1263	0.0786	0.1134
17 - 21	19.11	0.1189	-0.4579*	0.3968*
21 - 29	24.36	-0.6047*	0.2719□	-0.1641
29 - 50	35.96	0.1129	-0.3241*	-0.3456*

affected by the non-normality of the data. The χ^2-test, as used in this work, tests whether or not the observations fit a normal population in the sense that they could have arisen from it with any acceptable degree of probability. The statistic used is given by equation A.135. In the first classification of χ^2-tests all of the data in the shear system were considered simultaneously. The categories were 5 m/sec velocity intervals of the domain of the velocity component being tested. For example, in the case of the parallel component, category number 1 was the velocity interval -35 m/sec to -30 m/sec while category number 9 was the velocity interval + 5 m/sec to 10 m/sec. For this particular component, there were a total of 13 categories. The null hypothesis is that the observations arose from a normal population with a mean and standard deviation estimated from the observations. The probabilities π_i of an observation falling into each of these 13 categories were determined. When the probability π_i was multiplied by the number of observations in the sample (in the all data classification the number was 2245), one obtained the expected frequency of observations in the i th category based on the null hypothesis. The expected frequency was subtracted from the actual observed frequency in the i th category, and the difference after being squared was divided by the expected frequency. This was done for each category and the sum over all k categories gave the χ^2 statistic with k-3 degrees of freedom. For the parallel component, all data classification, k was 13 and the degrees of freedom were 10. Large discrepancies between the expected frequencies and the observed frequencies give rise to a large value for the χ^2 statistic and indicate the inadequacy of the null hypothesis. After a value for χ^2 has been calculated, a table for χ^2 is consulted (with reference to the proper number of degrees of freedom) to determine the probability that this value could have arisen by chance where the population conformed to the null hypothesis. If the probability is small, say at the 1% level or the 5% level, the null hypothesis is rejected. This procedure was performed for all three components in the all data classification. Next the observations were divided into the five shear magnitude classifications (the categories were five m/sec velocity intervals of the domain of the component being tested) and the χ^2 test was applied to each of the three velocity components in each of the five shear magnitude classifications. Then the observations were divided into the four altitude classifications (again the categories were five m/sec velocity intervals) and the χ^2 test was applied to each of the velocity components in each of the four altitude classifications. Table 12 gives the results of all of the aforementioned applications of the χ^2-test. An asterisk indicates that the probability of the null hypothesis being true is less than or equal to 1%, and a □ indicates that the probability of the null hypothesis being true is between only 1% and 5%. As is shown in the table, the χ^2-test indicates in almost every case that the null hypothesis should most definitely be rejected, thus giving firm confirmation to the non-Gaussian character of the observations.

After establishing the non-Gaussian character of the data, one then attempts to determine the factors which affect the variate. Now suppose, for the moment, that the population (from which the sample is

Table 12. Values of χ^2-test. (The number of degrees of freedom is abbreviated to d.f. An asterisk indicates a probability at the 1% level and a ◘ indicates a probability at the 5% level.)

Classification	Parallel Component		Perpendicular Component		Vertical Component	
	χ^2	d.f.	χ^2	d.f.	χ^2	d.f.
All Data	187.45*	10	222.06*	8	237.29*	10
Shear Magnitude						
0-12	44.97*	8	41.43*	7	41.20*	8
12-17	29.80*	8	44.61*	7	29.51*	8
17-21	25.89*	7	16.69*	5	26.19*	6
21-29	31.61*	8	10.49	6	10.16	7
29-50	16.93◘	9	25.66*	6	39.32*	7
Altitude						
90-95	19.58*	6	31.65*	6	14.67◘	7
95-100	24.42*	7	48.65*	5	35.86*	6
100-105	50.43*	10	48.53*	7	35.47*	8
105-110	55.64*	8	25.46*	7	39.40*	8

drawn) is homogeneous with respect to the factor of classification, that is, that the factor has no effect on the value of the variate. In this case if the population is divided into classes according to this factor, the different classes will have the same statistical properties. In particular the classes will have the same mean $\langle v \rangle$ and the same variance σ^2, which are the mean and variance of the population. On the other hand, if the population is non-homogeneous with respect to the factor of classification (that is the factor does affect the value of the variate), then, when the population is divided into classes based on this factor, the various classes will have different statistical properties. Recall that, in the case of turbulent velocity components, the mean value is zero by definition and all significantly non-zero values are attributed either to inaccuracy in the determination of the mean winds or anomolous upward motion. Thus, for turbulent velocity components, a variation of statistical properties will be indicated by different variances. If the observations were divided into only two classes, then the F-test would be used. However if the observations are divided into k classes ($k>2$), then the M-test, which is discussed in Appendix A, is used.

The first factor of classification to be considered is the shear magnitude. The observations were divided into five classes based on the magnitude of the shear vector. The classes were the same as those used in the previous t and χ^2 tests, so they extended over the intervals 0-12 m/sec/km, 12-17 m/sec/km, 17-21 m/sec/km, 21-29 m/sec/km, and 29-50 m/sec/km. An M-test was applied to each of the components of the velocity and the results are tabulated in Table 13. Recall

Table 13. Values of the M-test. (The number of degrees of freedom is abbreviated to d.f. An asterisk indicates a probability at the 1% level and a □ indicates a probability at the 5% level.)

Factor of Classification	Parallel Component of Velocity		Perpendicular Component of Velocity		Vertical Component of Velocity	
	M	d.f.	M	d.f.	M	d.f.
Shear Factor Only	17.85*	4	12.20□	4	43.44*	4
Altitude Factor Only	35.17*	3	54.27*	3	40.09*	3
2-Way Classification Shear Magnitude Factor						
90–95	5.66	4	11.94□	4	5.17	4
95–100	14.90*	4	23.56*	4	5.52	4
100–105	19.27*	4	50.60*	4	38.20*	4
105–110	13.76*	4	2.82	4	16.02*	4
2-Way Classification Altitude Factor						
0–12	39.47*	3	51.50*	3	21.23*	3
12–17	2.14	3	27.11*	3	24.15*	3
17–21	8.14□	3	15.63*	3	10.75□	3
21–29	13.22*	3	6.30	3	4.79	3
29–50	6.04	3	40.79*	3	12.64*	3

that the quantity M is distributed like χ^2 with k-1 degrees of freedom, where k is the number of classes. Table 13, under the heading 'Shear Magnitude Factor Only', gives the results of the first M-tests. It can be seen that the hypothesis of homogeneity with respect to the magnitude of the shear vector is definitely rejected for all components. Thus the tentative conclusion is reached that the magnitude of the shear vector does affect the value of the turbulent velocity components.

The second factor of classification is the altitude. The observations were divided into the same four classes based on altitude as were used in the previous t and χ^2 tests. The classes were 90-95 km, 95-100 km, 100-105 km, and 105-110 km. The application of an M-test for each velocity component gave the results listed in Table 13 under the heading 'Altitude Factor Only'. It is shown that the hypothesis of homogeneity with respect to altitude is definitely rejected for all three components. Now it should be understood that the altitude itself does not affect the velocity components, but rather the quantities which vary with altitude would affect the velocity components. Some of these quantities are the density, the temperature, and the pressure. Since measurements of these quantities were not made simultaneously with the measurements of the velocities, the observations cannot be classified according to each of the quantities individually. Instead, effects of these quantities appear in the altitude classification.

Table 6 indicates that there is a small increase in the average value of the magnitude of shear vector between each of the altitude classes. Thus, in order to separate and remove any interaction between altitude and magnitude of the shear vector, a two-way classification was devised. The first such two-way classification was designed to check on the homogeneity of the population with respect to the magnitude of the shear vector after any possible altitude interaction had been removed. The procedure in this two-way classification was to restrict the observations to be considered to one of the altitude classes, say 90-95 km; and then these observations were divided into the five shear magnitude classes. The M-test was applied for each of the velocity components. The next step in the procedure was to search through all of the observations and then restrict consideration to those observations falling in the next altitude class, 95-100 km. This new set of observations was divided into the five shear magnitude classes, and the M-test was applied for each velocity component. Successive steps in the procedure considered next the observations in the 100-105 km altitude interval and finally those in the 105-110 km interval. Each of these sets of data was then divided into the five shear magnitude classes, and the M-tests were applied. The results of this first two-way classification are tabulated in Table 13 under the heading '2-Way Classification, Shear Magnitude Factor'. It can be seen that, in three of the four cases for both the perpendicular component and the parallel component and in two of the four cases for the vertical component, the hypothesis of homogeneity with respect to the magnitude of the shear vector is rejected. Thus the shear vector does affect the turbulent velocity components even after the interaction with altitude has been removed.

The second two-way classification was designed to check on the homogeneity of the population with respect to the altitude after any possible shear magnitude interaction had been removed. The first step was to search through all of the observations and then restrict consideration to those which fell in the initial shear magnitude class, 0-12 m/sec/km. The resulting observations were divided into the four altitude classes, and the M-test was applied for each velocity component. Successive steps restricted observations to be considered to each of the remaining shear magnitude classes. The data falling into each class were then divided into the four altitude classes, and M-tests were applied for all three velocity components in each case. The findings of the second two-way classification are tabulated in Table 13 under the heading '2-Way Classification, Altitude Factor'. It is shown that, in four of the five cases for both the perpendicular and vertical components and in three of the five cases for the parallel component, the hypothesis of homogeneity with respect to altitude is rejected. Thus the quantities (i.e. density) do affect the turbulent velocity components even after the interaction with the shear magnitude has been removed.

Finally, a procedure to test the stationarity of the p.d.f.'s was devised. The null hypothesis was taken to be that the population was homogeneous with respect to the factor time. A two-way classification, which would check on the homogeneity of the population with respect to time after any shear magnitude interaction has been removed, was developed. The three time classes (denoted early, middle, and late) were as follows. For each rocket release, the early time

Table 14. Values of the M-test for Check on Stationarity. (The number of degrees of freedom is abbreviated to d.f. An asterisk indicates a probability at the 1% level and a □ indicates a probability at the 5% level.)

Factor of Classification	Parallel Component of Velocity		Perpendicular Component of Velocity		Vertical Component of Velocity	
	M	d.f.	M	d.f.	M	d.f.
2-Way Classification, Time Factor						
0-12	$6.486^{\square}$	2	$7.254^{\square}$	2	2.804	2
12-17	0.171	2	0.869	2	1.756	2
17-21	21.87*	2	0.684	2	$6.518^{\square}$	2
21-29	2.844	2	0.577	2	3.714	2
29-50	2.037	2	0.085	2	$7.668^{\square}$	2

class extended for 150 seconds from the time that the first globule was observed. The middle time class extended from the 150 second mark until 300 seconds after the first globule was observed. The late time class extended from the 300 second mark throughout the duration of the rocket release. As before, the first step was to search through all of the observations and then restrict consideration to those falling in the first shear magnitude interval. The resulting observations were divided into the three time classes, and the M-test was applied to each of the components. Again, successive steps restricted observations to be considered to each of the remaining shear magnitude classes, and the data falling into each class were divided into the three time classes. M-tests were applied for all three velocity components in each class, and the results are listed in Table 14. It can be seen that, in only two of the five cases for both the parallel and vertical components and in only one case for the perpendicular component the null hypothesis is rejected. It should be noted that, of the five cases which indicated rejection of the null hypothesis, only one case was significant at the 1% level and the other four were between the 1% and the 5% levels. So although the conditions in the atmosphere are certainly not stationary, over the relatively short time duration of the releases the assumption of stationarity has some validity.

Recent Laboratory Research

Frenkiel and Klebanoff [1967] have recently studied the turbulent flow field downstream of a grid in a wind tunnel. The turbulent field (previously assumed to be stationary in time and to exhibit a

Gaussian p.d.f. for the turbulent velocities) was found to have non-Gaussian p.d.f.'s. Only after the ad hoc modification of the Gaussian p.d.f.'s with the bivariate Hermite polynomials, were they able to achieve good agreement with the experimental measurements.

It is the author's speculation that the bars of the grid would generate a shear which could produce the non-Gaussian character of the p.d.f.'s. These turbulent velocities could then be convected downstream so that, even though no shear existed at the point of measurement, a non-Gaussian p.d.f. would be measured.

Welch and Tomme [1967] investigated the turbulent velocity component parallel to the flow (perpendicular to the shear) for a shear flow in a transparent pipe. This investigation concerned itself with the use of a laser velocimeter in determining the velocities in a pipe, but they did publish the p.d.f. of the velocity component parallel to the flow at several points not on the axis of the pipe so that the measurements were at points of non-zero shear magnitude. Calculations based on the published p.d.f. indicate a flatness factor which is higher than that associated with a Gaussian distribution and thereby confirm the non-Gaussian characteristics of the p.d.f. in shear turbulence. Since certain facts (such as number of data observations) were not published, exact statistical tests could not be made on the p.d.f. Also Welch and Tomme failed to simultaneously measure velocity components other than the one parallel to the flow, so the investigation of some elements of the theory in Chapter 3 could not be accomplished with their data.

CHAPTER III

THEORY

Introduction

Two lines of reasoning initially generated the suspicion that the shear vector would influence the p.d.f. for the turbulent velocities. The first involved consideration of equation 1.2. If the spatially dependent forces on the right hand side together with the last term on the left are labelled as F_i, then one has

$$\frac{\partial v_i}{\partial t} + V_j \frac{\partial v_i}{\partial x_j} + v_j \frac{\partial v_i}{\partial x_j}$$

$$= - v_j \frac{\partial V_i}{\partial x_j} + F_i \qquad (3.1)$$

where V_j is the j th component of the mean wind, v_j is the j th turbulent velocity component, and $\frac{\partial V_i}{\partial x_j}$ is the shear of the mean wind. For an incompressible fluid, the three terms on the left hand side of equation 3.1 represent the total derivative of v_i. The interesting term is the term which has been moved to the right hand side of equation 3.1, that is $-v_j \frac{\partial V_i}{\partial x_j}$. This term indicates that the derivative of the turbulent velocity v_i is proportional to what appears to be a velocity dependent force. The constant of proportionality is the negative of a component of the shear vector. The reasoning continued as follows: if the derivative of the velocity v_i is related to the shear vector, then it is possible for the p.d.f. to be related to the shear vector.

The second line of reasoning stems from research into the structure of turbulence by Justus [1968]. In his paper, Justus shows that, although many researchers have observed non-zero cross correlation coefficients of velocity components, theoretical considerations force the cross correlation coefficient of velocity components to be zero in both isotropic and axisymmetric turbulence. In fact, it is only when one considers symmetry about a plane that the cross correlation coefficient can be non-zero. The vectors chosen by Justus to define the plane of symmetry were the vertical direction and the direction parallel to the shear vector. Thus one reasons that the shear vector may have a basic significance in determining the properties of the turbulent velocity p.d.f., one moment of which is the correlation coefficient.

Based on these suspicions the analysis of variance statistical tests was performed. As it was shown in Chapter 2, the hypothesis of shear influence was completely justified since the population was found to be non-homogeneous with respect to the shear vector. Indeed the magnitude of the shear vector definitely affects the p.d.f. of the turbulent velocities.

At this point it should be noted that the experimental data which were presented represented, in effect, an ensemble average over the spatially-dependent portion of the p.d.f. The p.d.f. can be represented by a function which depends on both the spatial coordinates and the velocities. It is the velocity-dependent portion of the p.d.f. which will be the concern of this work; and except where indicated, it will be assumed that the spatially-dependent portion has been removed by integration over the domain of the spatial variables.

General Development

Phase space is a six-dimensional hyperspace with axes: x_1, x_2, x_3, v_1, v_2, and v_3. The position and turbulent velocities associated with a fluid element can be specified by giving its location in phase space.

Consider the probability density function $f(x_1,x_2,x_3,v_1,v_2,v_3,t)$. The quantity

$$f \quad dx_1dx_2dx_3dv_1dv_2dv_3$$

gives the probability of observing a fluid element at time t in the volume of phase space $dx_1dx_2dx_3dv_1dv_2dv_3$ around the point x_1,x_2,x_3,v_1,v_2,v_3. Stated another way, if a large number of observations were made, the above quantity would give the fractional number of observations for which a fluid element was found at time t to be in the volume of phase space $dx_1dx_2dx_3dv_1dv_2dv_3$ around the point x_1,x_2,x_3,v_1,v_2,v_3.

All of the fluid elements in a small volume $dv_1dv_2dv_3$ of velocity space (a subspace of phase space) located around the point v_1,v_2,v_3 are traveling with approximately the same velocity v (whose components are v_1,v_2,v_3). If one observed these fluid element throughout normal space at time t, there would be the fractional number $f\ dx_1dx_2dx_3dv_1dv_2dv_3$ of them in the volume $dx_1dx_2dx_3$ around the point x_1,x_2,x_3. Now assume that there does not occur any creation or destruction of physical fluid elements and assume that the fluid motions conform to the continuity equation (equation 1.1) and the slightly modified equation of fluctuating motion (equation 3.1). Then, the probability density $f(x_1,x_2,x_3,v_1,v_2,v_3,t)$ must equal the probability density $f(x_1',x_2',x_3',v_1',v_2',v_3',t')$, where

$$x_i' = x_i + \mu_{ij}v_j dt \tag{3.2}$$

$$v_i' = v_i + (F_i - \frac{\partial V_i}{\partial x_j} v_j)\ dt \tag{3.3}$$

$$t' = t + dt \tag{3.4}$$

where $i = 1,2,3$ and $j = 1,2,3$ and the Einstein summation convention is being used, and where μ_{ij} is a second order dimensionless tensor which allows the velocity components to be statistically coupled. Note that the only difference between equations 3.2, 3.3, and 3.4 and an equivalent set which would eventually provide a Gaussian distribution is the term in equation 3.3 which depends upon the shear components. So

$$f(x_i + \mu_{ij}v_j dt,\ v_i + (F_i - \frac{\partial V_i}{\partial x_j}\ v_j)\ dt,\ t + dt)$$

$$= f(x_i,\ v_i,\ t) \tag{3.5}$$

If one makes a Taylor series expansion about the point x_i,v_i,t of the left hand side of 3.5 and retains only terms in the first power of dt, one obtains the differential equation of the probability density function.

$$\frac{\partial f}{\partial t} + \mu_{ij} v_j \frac{\partial f}{\partial x_j} + (F_i - \frac{\partial V_i}{\partial x_j} v_j) \frac{\partial f}{\partial v_i} = 0 \tag{3.6}$$

Next one makes the assumption of stationarity for the p.d.f. This assumption is justified by the statistical test in Chapter II. Let

$$f(x_1,x_2,x_3,v_1,v_2,v_3) = f_x(x_1,x_2,x_3)\ f_v(v_1,v_2,v_3) \tag{3.7}$$

and recall that if Q is a function dependent upon spatial coordinates alone then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q(x_i)\ f_x\ dx_1 dx_2 dx_3 = < Q(x_i) > \tag{3.8}$$

and similarly if P is a function dependent upon velocity alone then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(v_i)\ f_v dv_1 dv_2 dv_3 = < P(v_i) > \tag{3.9}$$

where the < > indicates an expectation value. If one substitutes for f in equation 3.6 and recalls the stationarity, one has

$$\mu_{ij} v_j f_v \frac{\partial f_x}{\partial x_i} + (F_i - \frac{\partial V_i}{\partial x_j} v_j)\ f_x \frac{\partial f_v}{\partial v_i} = 0 \tag{3.10}$$

Now if one assumes that the spatially-dependent forces F_i can be represented as the derivative of a scalar potential

$$F_i = -\frac{\partial}{\partial x_i}\{\Omega(x_1,x_2,x_3)\} \qquad (3.11)$$

and that f_x is of the form

$$f_x = B_1 \exp\{-\beta\Omega(x_1,x_2,x_3)\} \qquad (3.12)$$

where β is a constant which serves the purpose of keeping the argument of the exponential function dimensionless (β has dimensions [energy/unit mass]$^{-1}$) and where B_1 is the normalizing factor for the spatially-dependent part of the p.d.f. (B_1 has dimensions $[m^3]^{-1}$), then upon substitution into 3.10 one obtains,

$$-\beta\,\mu_{ij}v_j f_v f_x \frac{\partial\Omega}{\partial x_i} - \left(\frac{\partial\Omega}{\partial x_i} + \frac{\partial V_i}{\partial x_j}v_j\right) f_x \frac{\partial f_v}{\partial v_i} = 0 \qquad (3.13)$$

As was pointed out earlier, the atmospheric data represents an average over the spatial coordinates. It should be noted that in a laboratory situation where one can exercise control over the spatial parameters, one could measure the spatially-dependent elements $\frac{\partial\Omega}{\partial x_i}$ and $\frac{\partial V_i}{\partial x_j}$ at the point in space where the velocity measurements were being made and then divide out the function f_x. In this way one would arrive at the velocity-dependent differential equation for the laboratory case. However as in the atmospheric case, if one averages equation 3.13 over the spatial coordinates, one has

$$-\beta\ \mu_{ij}\ v_j f_v \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_x\ \frac{\partial\Omega}{\partial x_i}\ dx_1 dx_2 dx_3$$

$$-\ \frac{\partial f_v}{\partial v_i} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_x\ \frac{\partial\Omega}{\partial x_i}\ dx_1 dx_2 dx_3$$

$$=\ v_j\ \frac{\partial f_v}{\partial v_i} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_x\ \frac{\partial V_i}{\partial x_j}\ dx_1 dx_2 dx_3 \qquad (3.14)$$

By equation 3.8, one obtains

$$-\ \beta\ \mu_{ij} v_j f_v\ \left< \frac{\partial\Omega}{\partial x_i} \right> - \left< \frac{\partial\Omega}{\partial x_i} \right> \frac{\partial f_v}{\partial v_i}$$

$$=\ \left< \frac{\partial V_i}{\partial x_j} \right> v_j\ \frac{\partial f_v}{\partial v_i} \qquad (3.15)$$

Let A_i equal the negative of the average value of the i th component of the spatially-dependent force, that is

$$A_i\ =\ -\ \left< F_i \right>\ =\ \left< \frac{\partial\Omega}{\partial x_i} \right> \qquad (3.16)$$

Consider the tensor $\left< \frac{\partial V_i}{\partial x_j} \right>$ and recall the discussion in Chapter II.

For the atmospheric case there are only two non-negligible and non-zero elements. So let

$$< \frac{\partial V_i}{\partial x_j} > = w_1\delta_{i1}\delta_{j3} + w_2\delta_{i2}\delta_{j3} \tag{3.17}$$

where

$$w_1 = < \frac{\partial V_1}{\partial x_3} > \tag{3.18}$$

and

$$w_2 = < \frac{\partial V_2}{\partial x_3} > \tag{3.19}$$

If, in order to simplify the notation, one adds to equation 3.17 the term $w_3\delta_{i3}\delta_{j3}$ where it is understood that

$$w_3 = 0 , \tag{3.20}$$

then equation 3.17 becomes

$$< \frac{\partial V_i}{\partial x_j} > = w_i\delta_{j3} \tag{3.21}$$

Substitution from 3.21 and 3.16 into 3.15 gives

$$-\beta\mu_{ij}v_j f_v A_i - A_i \frac{\partial f_v}{\partial v_i} = w_i v_3 \frac{\partial f_v}{\partial v_i} \tag{3.22}$$

Equation 3.22 is the differential equation for the velocity-dependent portion of the turbulent velocity probability density function, f_v. From now on f_v will be referred to as the velocity p.d.f. unless otherwise indicated. It should be noted that if the shear vector had zero amplitude (that is $w_i = 0$ for $i=1,2,3$) then equation 3.22 would give the Gaussian p.d.f. as the solution for f_v. Thus it is obvious that the introduction of a non-zero shear magnitude modifies the standard Gaussian solution.

Next one converts equation 3.22 into a dimensionless equation. This is done through the use of the following scale parameters: σ with dimensions m/sec, L with dimension m, ν with dimensions $\sec^{-1}$, and λ with dimensions $(m^3/sec^3)^{-1}$. Let

$$\hat{v}_i = \frac{v_i}{\sigma} \tag{3.23A}$$

$$\hat{x}_i = \frac{x_i}{L} \tag{3.23B}$$

$$\hat{F}_i = \frac{LF_i}{\sigma^2} = -\hat{A}_i = -\frac{LA_i}{\sigma^2} \tag{3.23C}$$

$$\hat{f}_v = \frac{f_v}{\lambda} \tag{3.23D}$$

$$\hat{\beta} = \beta\,\sigma^2 \tag{3.23E}$$

$$\hat{w}_i = \frac{w_i}{\nu} \tag{3.23F}$$

By substitution from equations 3.23 into equation 3.22, one obtains

$$-\hat{\beta}\ \mu_{ij}\hat{v}_j\hat{f}_v\hat{A}_i\ (\frac{\sigma\lambda}{L}) - \hat{A}_i\ \frac{\partial\hat{f}_v}{\partial\hat{v}_i}\ (\frac{\sigma\lambda}{L}) = (\lambda\nu)\hat{w}_i\hat{v}_3\ \frac{\partial\hat{f}_v}{\partial\hat{v}_i} \tag{3.24}$$

Division by $(\frac{\sigma\lambda}{L})$ gives

$$-\hat{\beta}\ \mu_{ij}\hat{v}_j\hat{f}_v\hat{A}_i - \hat{A}_i\ \frac{\partial\hat{f}_v}{\partial\hat{v}_i} = (\frac{L\nu}{\sigma})\ \hat{w}_i\hat{v}_3\ \frac{\partial\hat{f}_v}{\partial\hat{v}_i} \tag{3.25}$$

Let

$$\zeta = \frac{\nu L}{\sigma} \tag{3.26}$$

where ζ is a dimensionless perturbation parameter presumed to be small compared to 1. Then one has

$$-\hat{\beta}\ \mu_{ij}\hat{v}_j\hat{f}_v\hat{A}_i - \hat{A}_i\ \frac{\partial\hat{f}_v}{\partial\hat{v}_i} = \zeta\ \hat{w}_i\hat{v}_3\ \frac{\partial\hat{f}_v}{\partial\hat{v}_i} \tag{3.27}$$

The unperturbed solution will be given when ζ equals zero. If one assumes that

$$\hat{f}_v = \hat{f}_{v0}(1 + \zeta\phi_1 + \zeta^2\phi_2 + \zeta^3\phi_3 + \ldots) \tag{3.28}$$

where $\hat{f}_{v0}$ is the dimensionless unperturbed solution and where ϕ_n is the n th order correction to the unperturbed solution . (Note that the ϕ_n's are dimensionless), then if one substitutes from equation 3.28 into 3.27 and equates the terms of equal power of ζ^n, one obtains for n = 0

$$-\hat{\beta}\,\mu_{ij}\hat{v}_j\hat{f}_{v0}\hat{A}_i - \hat{A}_i\,\frac{\partial\hat{f}_{vo}}{\partial\hat{v}_i} = 0 \tag{3.29A}$$

for n = 1

$$-\hat{\beta}\,\mu_{ij}\hat{v}_j\hat{f}_{vo}\,\phi_1\hat{A}_i - \hat{A}_i\,\phi_1\,\frac{\partial\hat{f}_{vo}}{\partial\hat{v}_1}$$

$$-\hat{A}_i\hat{f}_{vo}\,\frac{\partial\phi_1}{\partial\hat{v}_i} = \hat{w}_i\hat{v}_3\,\frac{\partial\hat{f}_{vo}}{\partial\hat{v}_i} \tag{3.29B}$$

for n = 2

$$-\hat{\beta}\,\mu_{ij}\hat{v}_j\hat{f}_{vo}\,\phi_2\hat{A}_i - \hat{A}_i\,\phi_2\,\frac{\partial\hat{f}_{vo}}{\partial\hat{v}_i}$$

$$-\hat{A}_i\,\hat{f}_{vo}\,\frac{\partial\phi_2}{\partial\hat{v}_i} = \hat{w}_i\hat{v}_3\left(\phi_1\,\frac{\partial\hat{f}_{vo}}{\partial\hat{v}_i} + \hat{f}_{vo}\,\frac{\partial\phi_1}{\partial\hat{v}_i}\right) \tag{3.29C}$$

or for the general case n = k

$$-\hat{\beta}\,\mu_{ij}\hat{v}_j\hat{f}_{vo}\phi_k\hat{A}_i - \hat{A}_i\phi_k\,\frac{\partial\hat{f}_{vo}}{\partial\hat{v}_i}$$

$$-\hat{A}_i \hat{f}_{vo} \frac{\partial \phi_k}{\partial \hat{v}_i} = \hat{w}_i \hat{v}_3 \left(\phi_{k-1} \frac{\partial \hat{f}_{vo}}{\partial \hat{v}_i} + \hat{f}_{vo} \frac{\partial \phi_{k-1}}{\partial \hat{v}_i} \right) \tag{3.29D}$$

Rearrangement of equations 3.29 gives for n=0

$$\frac{\partial \hat{f}_{vo}}{\partial \hat{v}_i} + \hat{\beta} \mu_{ij} \hat{v}_j \hat{f}_{vo} = 0 \tag{3.30A}$$

for n = 1

$$\phi_1 \hat{A}_i \left(\frac{\partial \hat{f}_{vo}}{\partial \hat{v}_i} + \hat{\beta} \mu_{ij} \hat{v}_j \hat{f}_{vo} \right)$$

$$+ \hat{A}_i \hat{f}_{vo} \frac{\partial \phi_1}{\partial \hat{v}_i} = - \hat{w}_i \hat{v}_3 \frac{\partial \hat{f}_{vo}}{\partial \hat{v}_i} \tag{3.30B}$$

for n = 2

$$\phi_2 \hat{A}_i \left(\frac{\partial \hat{f}_{vo}}{\partial \hat{v}_i} + \hat{\beta} \mu_{ij} \hat{v}_j \hat{f}_{vo} \right)$$

$$+ \hat{A}_i \hat{f}_{vo} \frac{\partial \phi_2}{\partial \hat{v}_i} = - \hat{w}_i \hat{v}_3 \left(\phi_1 \frac{\partial \hat{f}_{vo}}{\partial \hat{v}_i} + \hat{f}_{vo} \frac{\partial \phi_1}{\partial \hat{v}_i} \right) \tag{3.30C}$$

or for the general case n = k

$$\phi_k \hat{A}_i \left(\frac{\partial \hat{f}_{vo}}{\partial \hat{v}_i} + \hat{\beta} \mu_{ij} \hat{v}_j \hat{f}_{vo} \right)$$

$$+ \hat{A}_i \hat{f}_{vo} \frac{\partial \phi_k}{\partial \hat{v}_i} = - \hat{w}_i \hat{v}_3 \left(\phi_{k-1} \frac{\partial \hat{f}_{vo}}{\partial \hat{v}_i} + \hat{f}_{vo} \frac{\partial \phi_{k-1}}{\partial \hat{v}_i} \right) \tag{3.30D}$$

It can be seen that the terms in the parenthesis on the left hand side of each equation for $n > 0$ are the same as those in the equation for $n = 0$ and therefore can be set equal to zero. If one makes use of this fact and also substitutes for $\frac{\partial \hat{f}_{vo}}{\partial \hat{v}_i}$ from equation 3.30A in each of the equations for $n > 0$, then after some slight rearrangement, one obtains

$$\frac{\partial \hat{f}_{vo}}{\partial \hat{v}_i} + \hat{\beta}\, \mu_{ij} \hat{v}_j \hat{f}_{vo} = 0 \tag{3.31A}$$

$$\hat{A}_i \frac{\partial \phi_1}{\partial \hat{v}_i} = \hat{\beta}\, \mu_{ij} \hat{w}_i \hat{v}_j \hat{v}_3 \tag{3.31B}$$

$$\hat{A}_i \frac{\partial \phi_2}{\partial \hat{v}_i} = \hat{w}_i \hat{v}_3 \left(\hat{\beta}\, \mu_{ij} \hat{v}_j \phi_1 - \frac{\partial \phi_1}{\partial \hat{v}_i} \right) \tag{3.31C}$$

$$\hat{A}_i \frac{\partial \phi_3}{\partial \hat{v}_i} = \hat{w}_i \hat{v}_3 \left(\hat{\beta}\, \mu_{ij} \hat{v}_j \phi_2 - \frac{\partial \phi_2}{\partial \hat{v}_i} \right) \tag{3.31D}$$

$$\vdots \qquad\qquad \vdots$$

$$\hat{A}_i \frac{\partial \phi_k}{\partial \hat{v}_i} = \hat{w}_i \hat{v}_3 \left(\hat{\beta}\, \mu_{ij} \hat{v}_j \phi_{k-1} - \frac{\partial \phi_{k-1}}{\partial \hat{v}_i} \right) \tag{3.31E}$$

$$\vdots \qquad\qquad \vdots$$

The series of equations given by 3.31 can be solved sequentially to arrive at the full solution for $\hat{f}_v$. The solution of equation 3.31A requires

$$\hat{f}_{vo} = B_2 \exp\left\{-\frac{\hat{\beta}}{2}\mu_{ij}\hat{v}_i\hat{v}_j\right\} \tag{3.32}$$

where B_2 is a dimensionless normalizing constant. Note that the solution for $\hat{f}_{vo}$ is a dimensionless trivariate Gaussian p.d.f.

At this point it is convenient to introduce the dimensions back into the equations and solutions. Thus in dimensional form, one would have

$$f_v = f_{vo}(1 + \psi_1 + \psi_3 + \ldots) \tag{3.33}$$

where

$$\psi_n = \zeta^n \phi_n \tag{3.34}$$

so ψ_n is dimensionless, and where f_{vo} is the trivariate Gaussian p.d.f. which has dimensions, $(m/sec)^{-3}$. Upon introduction of the dimensions, equation 3.32 becomes

$$f_{vo} = B_3 \exp\left\{-\frac{\beta}{2}\mu_{ij}v_iv_j\right\} \tag{3.35}$$

where B_3 is a normalizing constant with dimensions, $(m/sec)^{-3}$. Since the unperturbed solution is the trivariate Gaussian p.d.f., the notation is simplified and becomes identical with that of Appendices A and

F if one lets

$$B_3 = [(2\pi)^3 K \sigma_1{}^2\sigma_2{}^2\sigma_3{}^2]^{-1/2} \tag{3.36}$$

(Note that B_3 has the correct dimensions) and

$$\beta \mu_{ij} = a_{ij} \tag{3.37}$$

where a_{ij} represents the elements of the tensor $\overleftrightarrow{A}$ given in equation A.88 which also has the correct dimensions. So

$$f_{vo} = [(2\pi)^3 K \ \sigma_1{}^2\sigma_2{}^2\sigma_3{}^2]^{-1/2} \ \exp\{-\tfrac{1}{2} a_{ij}v_iv_j\} \tag{3.38}$$

Consider the introduction of dimensions into equation 3.31B, from equations 3.23 one has

$$\left(\frac{L}{\sigma^2}\right) A_i \ \sigma \ \frac{\partial\phi_1}{\partial v_i} = \sigma^2\beta\mu_{ij} \ \frac{w_i}{\nu} \frac{v_j}{\sigma} \frac{v_3}{\sigma}$$

Substitution from equation 3.37 and rearrangement gives

$$A_i\left(\frac{L\nu}{\sigma}\right) \frac{\partial\phi_1}{\partial v_i} = a_{ij}w_iv_jv_3$$

From equations 3.26 and 3.34, the above is seen to be simply

$$A_i \frac{\partial\psi_1}{\partial v_i} = a_{ij}w_iv_jv_3 \tag{3.39A}$$

In a similar manner, one finds

$$A_i \frac{\partial\psi_2}{\partial v_i} = w_i v_3 (a_{ij} v_j \psi_1 - \frac{\partial\psi_1}{\partial v_i}) \quad (3.39B)$$

$$A_i \frac{\partial\psi_3}{\partial v_i} = w_i v_3 (a_{ij} v_j \psi_2 - \frac{\partial\psi_2}{\partial v_i}) \quad (3.39C)$$

$$\vdots \qquad\qquad \vdots$$

$$A_i \frac{\partial\psi_n}{\partial v_i} = w_i v_3 (a_{ij} v_j \psi_{n-1} - \frac{\partial\psi_{n-1}}{\partial v_i}) \quad (3.39D)$$

$$\vdots \qquad\qquad \vdots$$

Thus the complete solution for f_v is given by equation 3.33 where f_{vo} is given in equation 3.38 and the ψ's can be found by the sequential solution of equations 3.39.

In the present work only the solution to the first-order perturbation equation will be considered. In other words, the approximation will be made that

$$f_v = f_{vo} (1 + \psi_1) \quad (3.40)$$

where ψ_1 is the solution of equation 3.39 A.

Solution of First-Order Perturbation Equation

Assume a series solution for ψ_1 of the form

$$\psi_1 = \sum_{i,j,k=0}^{\infty} A_{i,j,k} H_{i,j,k}(v_1, v_2, v_3) \tag{3.41}$$

where the $A_{i,j,k}$ are constants (which will be determined so that ψ_1 satisfies the differential equation 3.39A and the integral conditions that apply to f_v) and where the $H_{i,j,k}$ are the trivariate Hermite polynomials defined in Appendix F.

If one substitutes from equation 3.41 into equation 3.39A and recalls equations F.87, F.88, and F.89, then one has

$$A_1 \left(\sum_{i,j,k=0}^{\infty} A_{i,j,k}[i a_{11} H_{i-1,j,k} + j\ a_{12} H_{i,j-1,k} + k\ a_{13} H_{i,j,k-1}] \right)$$

$$+ A_2 \left(\sum_{i,j,k=0}^{\infty} A_{i,j,k}[i\ a_{12} H_{i-1,j,k} + j\ a_{22} H_{i,j-1,k} + k\ a_{23} H_{i,j,k-1}] \right)$$

$$+ A_3 \left(\sum_{i,j,k=0}^{\infty} A_{i,j,k}[i\ a_{13} H_{i-1,j,k} + j\ a_{23} H_{i,j-1,k} + k\ a_{33} H_{i,j,k-1}] \right)$$

$$= w_1 a_{11} v_3 v_1 + w_1 a_{12} v_3 v_2 + w_1 a_{13} v_3^{\,2}$$

$$+ w_2 a_{12} v_3 v_1 + w_2 a_{22} v_3 v_2 + w_2 a_{23} v_3^{\,2} \tag{3.42}$$

Now if one substitutes for $v_1 v_3$, $v_2 v_3$, and $v_3^{\,2}$ from equation F.105, F.106, and F.107 respectively and then equates the coefficients of identical $H_{i,j,k}$ polynomials, one obtains the following relations:

For H_{000},

$$A_1(A_{100}a_{11} + A_{010}a_{12} + A_{001}a_{13}) + A_2(A_{100}a_{12} + A_{010}a_{22} + A_{001}a_{23})$$

$$+ A_3(A_{100}a_{13} + A_{010}a_{23} + A_{001}a_{33}) = (w_1a_{11} + w_2a_{21})(\sigma_1\sigma_3\sigma_{13})$$

$$+ (w_1a_{12} + w_2a_{22})(\sigma_2\sigma_3\sigma_{23}) + (w_1a_{13} + w_2a_{23})\sigma_3^{\;2} \tag{3.43}$$

For H_{100},

$$A_1(A_{110}a_{12} + A_{101}a_{13} + 2\ A_{200}a_{11}) + A_2(A_{110}a_{22} + A_{101}a_{23} + 2\ A_{200}a_{12})$$

$$+ A_3(A_{110}a_{23} + A_{101}a_{33} + 2\ A_{200}a_{13}) = 0 \tag{3.44}$$

For H_{010},

$$A_1(A_{011}a_{13} + 2\ A_{020}a_{12} + A_{110}a_{11}) + A_2(A_{110}a_{12} + A_{011}a_{23} + 2\ A_{020}a_{22})$$

$$+ A_3(A_{110}a_{13} + A_{011}a_{33} + 2\ A_{020}a_{23}) = 0 \tag{3.45}$$

For H_{001},

$$A_1(A_{101}a_{11} + A_{011}a_{12} + 2\ A_{002}a_{13}) + A_2(A_{101}a_{12} + A_{011}a_{22} + 2\ A_{002}a_{23})$$

$$+ A_3(A_{101}a_{13} + A_{011}a_{23} + 2\ A_{002}a_{33}) = 0 \tag{3.46}$$

For H_{110},

$$A_1(A_{111}a_{13} + 2\ A_{120}a_{12} + 2\ A_{210}a_{11}) + A_2(A_{111}a_{23} + 2\ A_{120}a_{22} + 2\ A_{210}a_{12})$$

$$+ A_3(A_{111}a_{33} + 2\ A_{120}a_{23} + 2\ A_{210}a_{13}) = (w_1a_{11} + w_2a_{21})(\sigma_1{}^2\sigma_2\sigma_3[\rho_{23}$$

$$+ \rho_{12}\rho_{13}]) + (w_1a_{12} + w_2a_{22})(\sigma_1\sigma_2{}^2\sigma_3[\rho_{13} + \rho_{12}\rho_{23}]) + (w_1a_{13}$$

$$+ w_2a_{23})(2\ \sigma_1\sigma_2\sigma_3{}^2\ \rho_{13}\rho_{23}) \qquad (3.47)$$

For H_{101},

$$A_1(A_{111}a_{12} + 2\ A_{102}a_{13} + 2\ A_{201}a_{11}) + A_2(A_{111}a_{22} + 2\ A_{102}a_{23} + 2\ A_{201}a_{12})$$

$$+ A_3(A_{111}a_{23} + 2\ A_{102}a_{33} + 2\ A_{201}a_{13}) = (w_1a_{11} + w_2a_{21})(\sigma_1{}^2\sigma_3{}^2[1$$

$$+ \rho_{13}{}^2]) + (w_1a_{12} + w_2a_{22})(\sigma_1\sigma_2\sigma_3{}^2[\rho_{12} + \rho_{13}\rho_{23}]) + (w_1a_{13}$$

$$+ w_2a_{23})(2\ \sigma_1\sigma_3{}^3\rho_{13}) \qquad (3.48)$$

For H_{011},

$$A_1(A_{111}a_{11} + A_{012}\ 2\ a_{13} + 2\ A_{021}a_{12}) + A_2(A_{111}a_{12} + 2\ A_{012}a_{23}$$

$$+ 2\ A_{021}a_{22}) + A_3(A_{111}a_{13} + 2\ A_{012}a_{33} + 2\ A_{021}a_{23}) = (w_1a_{11}$$

$$+ w_2a_{21})(\sigma_1\sigma_2\sigma_3^2 [\rho_{12} + \rho_{13}\rho_{23}]) + (w_1a_{12} + w_2a_{22})(\sigma_2^2\sigma_3^2 [1$$

$$+ \rho_{23}^2]) + (w_1a_{13} + w_2a_{23})(2 \sigma_2\sigma_3^3\rho_{23}) \tag{3.49}$$

For H_{200},

$$A_1(A_{210}a_{12} + A_{201}a_{13} + 3 A_{300}a_{11}) + A_2(A_{210}a_{22} + A_{201}a_{23} + 3 A_{300}a_{12})$$

$$+ A_3(A_{210}a_{23} + A_{201}a_{33} + 3 A_{300}a_{13}) = (w_1a_{11} + w_2a_{21})(\sigma_1^3\sigma_3\rho_{13})$$

$$+ (w_1a_{12} + w_2a_{22})(\sigma_1^2\sigma_2\sigma_3\rho_{12}\rho_{13}) + (w_1a_{13} + w_2a_{23})(2 \sigma_1^2\sigma_3^2\rho_{13}^2) \tag{3.50}$$

For H_{020}

$$A_1(A_{120}a_{11} + A_{021}a_{13} + 3 A_{030}a_{12}) + A_2(A_{120}a_{12} + A_{021}a_{23} + 3 A_{030}a_{22})$$

$$+ A_3(A_{120}a_{13} + A_{021}a_{33} + 3 A_{030}a_{23}) = (w_1a_{11} + w_2a_{21})(\sigma_1\sigma_2^2\sigma_3\rho_{12}\rho_{23})$$

$$+ (w_1a_{12} + w_2a_{22})(\sigma_2^3\sigma_3\rho_{23}) +(w_1a_{13} + w_2a_{23})(2 \sigma_2^2\sigma_3^2\rho_{23}^2) \tag{3.51}$$

For H_{002}

$$A_1(A_{102}a_{11} + A_{012}a_{12} + 3 A_{003}a_{13}) + A_2(A_{102}a_{12} + A_{012}a_{22} + 3 A_{003}a_{23})$$

$$+ A_3(A_{102}a_{13} + A_{012}a_{23} + 3 A_{003}a_{33}) = (w_1a_{11} + w_2a_{21})(\sigma_1\sigma_3^3\rho_{13})$$

$$+ (w_1 a_{12} + w_2 a_{22})(\sigma_2 \sigma_3^3 \rho_{23}) + (w_1 a_{13} + w_2 a_{23})(2\ \sigma_3^4) \tag{3.52}$$

For H_{210},

$$A_1(2\ A_{220}a_{12} + A_{211}a_{13} + 3\ A_{310}a_{11}) + A_2(2\ A_{220}a_{22} + A_{211}a_{23}$$

$$+ 3\ A_{310}a_{12}) + A_3(2\ A_{220}a_{23} + A_{211}a_{33} + 3\ A_{310}a_{13}) = 0 \tag{3.53}$$

For H_{201},

$$A_1(2\ A_{202}a_{13} + A_{211}a_{12} + 3\ A_{301}a_{11}) + A_2(2\ A_{202}a_{23} + A_{211}a_{22}$$

$$+ 3\ A_{301}a_{12}) + A_3(2\ A_{202}a_{33} + A_{211}a_{23} + 3\ A_{301}a_{13}) = 0 \tag{3.54}$$

For H_{120},

$$A_1(2\ A_{220}a_{11} + A_{121}a_{13} + 3\ A_{130}a_{12}) + A_2(2\ A_{220}a_{12} + A_{121}a_{23}$$

$$+ 3\ A_{130}a_{22}) + A_3(2\ A_{220}a_{13} + A_{121}a_{33} + 3\ A_{130}a_{23}) = 0 \tag{3.55}$$

For H_{102},

$$A_1(A_{112}a_{12} + 2\ A_{202}a_{11} + 3\ A_{103}a_{13}) + A_2(A_{112}a_{22} + 2\ A_{202}a_{12}$$

$$+ 3\ A_{103}a_{23}) + A_3(A_{112}a_{23} + 2\ A_{202}a_{13} + 3\ A_{103}a_{33}) = 0 \tag{3.56}$$

For H_{021},

$$A_1(A_{121}a_{11} + 2\ A_{022}a_{13} + 3\ A_{031}a_{12}) + A_2(A_{121}a_{12} + 2\ A_{022}a_{23}$$

$$+ 3\ A_{031}a_{22}) + A_3(A_{121}a_{13} + 2\ A_{022}a_{33} + 3\ A_{031}a_{23}) = 0 \qquad (3.57)$$

For H_{012},

$$A_1(A_{112}a_{11} + 2\ A_{022}a_{12} + 3\ A_{013}a_{13}) + A_2(A_{112}a_{12} + 2\ A_{022}a_{22}$$

$$+ 3\ A_{013}a_{23}) + A_3(A_{112}a_{13} + 2\ A_{022}a_{23} + 3\ A_{013}a_{33}) = 0 \qquad (3.58)$$

For H_{111},

$$A_1(2\ A_{112}a_{13} + 2\ A_{121}a_{12} + 2\ A_{211}a_{11}) + A_2(2\ A_{112}a_{23} + 2\ A_{121}a_{22}$$

$$+ 2\ A_{211}a_{12}) + A_3(2\ A_{112}a_{33} + 2\ A_{121}a_{23} + 2\ A_{211}a_{13}) = 0 \qquad (3.59)$$

For H_{300},

$$A_1(A_{310}a_{12} + A_{301}a_{13} + 4\ A_{400}a_{11}) + A_2(A_{310}a_{22} + A_{301}a_{23}$$

$$+ 4\ A_{400}a_{12}) + A_3(A_{310}a_{23} + A_{301}a_{33} + 4\ A_{400}a_{13}) = 0 \qquad (3.60)$$

For H_{030},

$$A_1(A_{130}a_{11} + A_{031}a_{13} + 4\ A_{040}a_{12}) + A_2(A_{130}a_{12} + A_{031}a_{23}$$

$$+ 4\ A_{040}a_{22}) + A_3(A_{130}a_{13} + A_{031}a_{33} + 4\ A_{040}a_{23}) = 0 \qquad (3.61)$$

For H_{003},

$$A_1(A_{103}a_{11} + A_{013}a_{12} + 4\ A_{004}a_{13}) + A_2(A_{103}a_{12} + A_{013}a_{22}$$

$$+ 4\ A_{004}a_{23}) + A_3(A_{103}a_{13} + A_{013}a_{23} + 4\ A_{004}a_{33}) = 0 \qquad (3.62)$$

For $H_{\ell,m,n}$, where $\ell + m + n \geq 3$, one has

$$A_1([\ell+1]a_{11}A_{\ell+1,m,n} + [m+1]\ a_{12}A_{\ell,m+1,n} + [n+1]\ a_{13}A_{\ell,m,n+1})$$

$$+ A_2([\ell+1]a_{12}A_{\ell+1,m,n} + [m+1]\ a_{22}A_{\ell,m+1,n} + [n+1]\ a_{23}A_{\ell,m,n+1})$$

$$+ A_3([\ell+1]a_{13}A_{\ell+1,m,n} + [m+1]\ a_{23}A_{\ell,m+1,n} + [n+1]\ a_{33}A_{\ell,m,n+1})$$

$$= 0 \qquad (3.63)$$

Equations 3.42 through 3.63 provide the restrictions placed on the $A_{i,j,k}$ coefficients by the differential equation. In this sense, they can be called "constraint" equations on the $A_{i,j,k}$ coefficients. Let the order of the "constraint" equation be denoted by the integer equal to the sum of the subscripts on the Hermite polynomial whose coefficients were set equal in order to arrive at the "constraint" equation. For example, equation 3.53 would be of order 3 since the coefficients of $H_{2,1,0}$ were set equal in order to arrive at equation 3.53.

We note that equations 3.44, 3.45 and 3.46 are all of the same order and that certain $A_{\ell,m,n}$ coefficients appear in two of the three equations (for example A_{110} and A_{101}). Note also that in general an $A_{\ell,m,n}$ appears only in those equations of order $\ell + m + n - 1$. Since equations of different order have no unknowns ($A_{\ell mn}$) in common, each order may be treated separately. However, for a given order, all of the equations of that order must be treated as a set of simultaneous equations for the $A_{\ell mn}$ coefficients of that order.

In general, there will be

$$\frac{(p+1)(p+2)}{2}$$

equations of order p, but there will be

$$\frac{(p+2)(p+3)}{2}$$

unknowns ($A_{\ell,m,n}$) in the equations of order p. Since the number of equations is less than the number of unknowns for each order, the simultaneous set of equations for each order will have infinitely many solutions. There will be p + 2 arbitrary constants involved in the solution for the $A_{\ell mn}$ coefficients of order p. Thus the restrictions placed on the $A_{\ell,m,n}$ coefficients by the differential equation can certainly be satisfied.

One may gain a better feeling for the $A_{\ell,m,n}$ coefficients by determining these coefficients in terms of moments about the mean or in terms of cumulants. If one substitutes for ψ_1 from equation 3.41 into equation 3.40, one obtains

$$f_v(v_1,v_2,v_3) = f_{vo}[H_{o,o,o} + \sum_{i,j,k=0}^{\infty} A_{i,j,k}H_{i,j,k}(v_1,v_2,v_3)] \tag{3.64}$$

Now multiplication of both sides of equation 3.64 by the complementary Hermite polynomials $G_{\ell,m,n}$ (see Appendix F) and integration over all velocity space gives

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_{\ell,m,n}f_v dv_1dv_2dv_3 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{vo}G_{\ell,m,n}[H_{0,0,0}$$

$$+ \sum_{i,j,k=0}^{\infty} A_{i,j,k}H_{i,j,k}(v_1,v_2,v_3)]dv_1dv_2dv_3 \tag{3.65}$$

By the orthogonality relation F.73, one obtains for ℓ=m=n=0

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 1\cdot f_v dv_1dv_2dv_3 = 1 + A_{0,0,0} \tag{3.66}$$

By equation 3.9, the left hand side of 3.66 is equal to 1. Therefore one has

$$A_{0,0,0} = 0 \tag{3.67}$$

For $\ell + m + n > 0$, by equation F.73, one obtains

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_{\ell,m,n} f_v dv_1 dv_2 dv_3 = A_{\ell,m,n}\, \ell! m! n! \tag{3.68}$$

But by equation 3.9, the left hand side of 3.68 is equal to $\langle G_{\ell,m,n}\rangle$.

Thus one has for $\ell + m + n > 0$

$$A_{\ell,m,n} = (\ell! m! n!)^{-1} \langle G_{\ell,m,n}\rangle \tag{3.69}$$

Consider the case ℓ=1, m=0, n=0. From F.100 one has

$$G_{1,0,0} = v_1$$

But recall that by definition, $\langle v_1\rangle = 0$. Thus one obtains

$$A_{1,0,0} = 0 \tag{3.70}$$

In a similar manner one finds

$$A_{0,1,0} = A_{0,0,1} = 0 \tag{3.71}$$

Consider the case ℓ=2, m=0, n=0. From F.100 one has

$$G_{2,0,0} = v_1^{\,2} - \sigma_1^{\,2}$$

Since the mean value is zero, $<v_1^{\,2}>$ is equal to $\mu_{2,0,0}$. But by definition, $\mu_{2,0,0} = \sigma_1^{\,2}$. Thus one obtains

$$A_{2,0,0} = \frac{1}{2} \; (<v_1^{\,2}> - \sigma_1^{\,2})$$

$$A_{2,0,0} = \frac{1}{2} \; (\mu_{200} - \sigma_1^{\,2}) = 0 \tag{3.72}$$

In a similar manner one finds the following results for the $A_{\ell,m,n}$ for $\ell + m + n \leq 4$:

$$A_{\ell,m,n} = 0 \text{ for } \ell + m + n \leq 2$$

$$A_{300} = \frac{1}{6} \; \mu_{300} \qquad A_{030} = \frac{1}{6} \; \mu_{030} \qquad A_{003} = \frac{1}{6} \; \mu_{003}$$

$$A_{210} = \frac{1}{2} \; \mu_{210} \qquad A_{120} = \frac{1}{2} \; \mu_{120} \qquad A_{201} = \frac{1}{2} \; \mu_{201}$$

$$A_{102} = \frac{1}{2} \; \mu_{102} \qquad A_{021} = \frac{1}{2} \; \mu_{021} \qquad A_{012} = \frac{1}{2} \; \mu_{012}$$

$$A_{111} = \mu_{111} \qquad A_{400} = \frac{1}{24} (\mu_{400} - 3 \, \sigma_1^{\,4}) \tag{3.73}$$

$$A_{040} = \frac{1}{24} (\mu_{040} - 3 \, \sigma_2^{\,4}) \qquad A_{004} = \frac{1}{24} (\mu_{004} - 3 \, \sigma_3^{\,4})$$

$$A_{301} = \frac{1}{6} \; (\mu_{301} - 3 \, \rho_{13}\sigma_1^{\,3}\sigma_3)$$

$$A_{103} = \frac{1}{6} (\mu_{103} - 3\ \rho_{13}\sigma_1\sigma_3{}^3)$$

$$A_{310} = \frac{1}{6} (\mu_{310} - 3\ \rho_{12}\sigma_1{}^3\sigma_2)$$

$$A_{130} = \frac{1}{6} (\mu_{130} - 3\ \rho_{12}\sigma_1\sigma_2{}^3)$$

$$A_{031} = \frac{1}{6} (\mu_{031} - 3\ \rho_{23}\sigma_2{}^3\sigma_3)$$

$$A_{013} = \frac{1}{6} (\mu_{013} - 3\ \rho_{23}\sigma_2\sigma_3{}^3)$$

$$A_{220} = \frac{1}{4} (\mu_{220} - \sigma_1{}^2\sigma_2{}^2 - 2\ \rho_{12}{}^2\sigma_1{}^2\sigma_2{}^2)$$

$$A_{202} = \frac{1}{4} (\mu_{202} - \sigma_1{}^2\sigma_2{}^2 - 2\ \rho_{13}{}^2\sigma_1{}^2\sigma_3{}^2) \qquad \text{(3.73 Cont)}$$

$$A_{022} = \frac{1}{4} (\mu_{022} - \sigma_2{}^2\sigma_3{}^2 - 2\ \rho_{23}{}^2\sigma_2{}^2\sigma_3{}^2)$$

$$A_{211} = \frac{1}{2} (\mu_{211} - \rho_{23}\sigma_1{}^2\sigma_2\sigma_3 - 2[\rho_{12}\sigma_1\sigma_2][\rho_{13}\sigma_1\sigma_3])$$

$$A_{121} = \frac{1}{2} (\mu_{121} - \rho_{13}\sigma_1\sigma_2{}^2\sigma_3 - 2[\rho_{12}\sigma_1\sigma_2][\rho_{23}\sigma_2\sigma_3])$$

$$A_{112} = \frac{1}{2} (\mu_{112} - \rho_{12}\sigma_1\sigma_2\sigma_3{}^2 - 2[\rho_{13}\sigma_1\sigma_3][\rho_{23}\sigma_2\sigma_3])$$

If one substitutes for the moments about the mean in terms of the cumulants from equation A.54, one obtains for $A_{\ell,m,n}$ where $2<\ell+m+n\leq 4$ the following:

$$A_{\ell,m,n} = (\ell!m!n!)^{-1} \kappa_{\ell,m,n} \text{ for } 2 < \ell + m + n \leq 4 \tag{3.74}$$

Now if one rewrites the "constraint" equations of order 2 or less and substitutes for the $A_{\ell,m,n}$ coefficients from equation 3.73 and substitutes for the a_{ij} from A.88, one obtains:

For order 0,

$$0 = \frac{w_1\sigma_3}{\sigma_1 K}(\rho_{13} - \rho_{13}\rho_{23}^2 - \rho_{12}\rho_{23} + \rho_{13}\rho_{23}^2 - \rho_{13} + \rho_{12}\rho_{23})$$

$$+ \frac{w_2\sigma_3}{\sigma_2 K}(-\rho_{12}\rho_{13} + \rho_{13}^2\rho_{23} - \rho_{13}^2\rho_{23} - \rho_{23} + \rho_{23} + \rho_{12}\rho_{13})$$

$$0 = 0 \tag{3.75}$$

For order 1,

$$0 = 0 \tag{3.76}$$

For order 2,

$$A_1(\mu_{111}\left[-\frac{\rho_{13}-\rho_{12}\rho_{23}}{K\sigma_1\sigma_3}\right] + \mu_{120}\left[-\frac{\rho_{12}-\rho_{13}\rho_{23}}{K\sigma_1\sigma_2}\right] + \mu_{210}\left[\frac{(1-\rho_{23}^2)}{K\sigma_1^2}\right])$$

$$+ A_2(\mu_{111}\left[-\frac{\rho_{23}-\rho_{12}\rho_{13}}{K\sigma_2\sigma_3}\right] + \mu_{120}\left[\frac{(1-\rho_{13}^2)}{K\sigma_2^2}\right] + \mu_{210}\left[-\frac{\rho_{12}-\rho_{13}\rho_{23}}{K\sigma_1\sigma_2}\right]) \tag{3.77}$$

$$+ A_3(\mu_{111}[\frac{(1-\rho_{12}^2)}{K\sigma_3^2}] + \mu_{120}[-\frac{\rho_{23}-\rho_{12}\rho_{13}}{K\sigma_2\sigma_3}] + \mu_{210}[-\frac{\rho_{13}-\rho_{12}\rho_{23}}{K\sigma_1\sigma_3}])$$

$$= w_1\rho_{23}\sigma_2\sigma_3 + w_2\rho_{13}\sigma_1\sigma_3$$

$$A_1(\mu_{111}[-\frac{\rho_{12}-\rho_{13}\rho_{23}}{K\sigma_1\sigma_2}] + \mu_{102}[-\frac{\rho_{13}-\rho_{12}\rho_{23}}{K\sigma_1\sigma_3}] + \mu_{201}[\frac{1-\rho_{23}^2}{K\sigma_1^2}])$$

$$+ A_2(\mu_{111}[\frac{1-\rho_{13}^2}{K\sigma_2^2}] + \mu_{102}[-\frac{\rho_{23}-\rho_{12}\rho_{13}}{K\sigma_2\sigma_3}] + \mu_{201}[-\frac{\rho_{12}-\rho_{13}\rho_{23}}{K\sigma_1\sigma_2}])$$

(3.77 Cont)

$$+ A_3(\mu_{111}[-\frac{\rho_{23}-\rho_{12}\rho_{13}}{K\sigma_2\sigma_3}] + \mu_{102}[\frac{1-\rho_{12}^2}{K\sigma_3^2}] + \mu_{201}[-\frac{\rho_{13}-\rho_{12}\rho_{23}}{K\sigma_1\sigma_3}]) = w_1\sigma_3^2$$

$$A_1(\mu_{111}[\frac{1-\rho_{23}^2}{K\sigma_1^2}] + \mu_{012}[-\frac{\rho_{13}-\rho_{12}\rho_{23}}{K\sigma_1\sigma_3}] + \mu_{021}[-\frac{\rho_{12}-\rho_{13}\rho_{23}}{K\sigma_1\sigma_2}])$$

$$+ A_2(\mu_{111}[-\frac{\rho_{12}-\rho_{13}\rho_{23}}{K\sigma_1\sigma_2}] + \mu_{012}[-\frac{\rho_{23}-\rho_{12}\rho_{13}}{K\sigma_2\sigma_3}] + \mu_{021}[\frac{1-\rho_{13}^2}{K\sigma_2^2}])$$

$$+ A_3(\mu_{111}[-\frac{\rho_{13}-\rho_{12}\rho_{23}}{K\sigma_1\sigma_3}] + \mu_{012}[\frac{1-\rho_{12}^2}{K\sigma_3^2}] + \mu_{021}[-\frac{\rho_{23}-\rho_{12}\rho_{13}}{K\sigma_2\sigma_3}]) = w_2\sigma_3^2$$

$$\frac{A_1}{2}\left(\mu_{210}\left[-\frac{\rho_{12}-\rho_{13}\rho_{23}}{K\sigma_1\sigma_2}\right] + \mu_{201}\left[-\frac{\rho_{13}-\rho_{12}\rho_{23}}{K\sigma_1\sigma_3}\right] + \mu_{300}\left[\frac{1-\rho_{23}^{2}}{K\sigma_1^{2}}\right]\right)$$

$$+ \frac{A_2}{2}\left(\mu_{210}\left[\frac{1-\rho_{13}^{2}}{K\sigma_2^{3}}\right] + \mu_{201}\left[-\frac{\rho_{23}-\rho_{12}\rho_{13}}{K\sigma_2\sigma_3}\right] + \mu_{300}\left[-\frac{\rho_{12}-\rho_{13}\rho_{23}}{K\sigma_1\sigma_2}\right]\right)$$

$$+ \frac{A_3}{2}\left(\mu_{210}\left[-\frac{\rho_{23}-\rho_{12}\rho_{13}}{K\sigma_2\sigma_3}\right] + \mu_{201}\left[\frac{1-\rho_{12}^{2}}{K\sigma_3^{2}}\right] + \mu_{300}\left[-\frac{\rho_{13}-\rho_{12}\rho_{23}}{K\sigma_1\sigma_3}\right]\right)$$

$$= w_1\rho_{13}\sigma_1\sigma_3\left(1 - \frac{\rho_{13}}{K}\left[\rho_{13}-\rho_{12}\rho_{23}\right]\right) + w_2\frac{\sigma_1^{2}\sigma_3}{K\sigma_2}\rho_{13}^{2}(\rho_{12}\rho_{13}-\rho_{23})$$

(3.77 Cont)

$$\frac{A_1}{2}\left(\mu_{120}\left[\frac{1-\rho_{23}^{2}}{K\sigma_1^{2}}\right] + \mu_{021}\left[-\frac{\rho_{13}-\rho_{12}\rho_{23}}{K\sigma_1\sigma_3}\right] + \mu_{030}\left[-\frac{\rho_{12}-\rho_{13}\rho_{23}}{K\sigma_1\sigma_2}\right]\right)$$

$$+ \frac{A_2}{2}\left(\mu_{120}\left[-\frac{\rho_{12}-\rho_{13}\rho_{23}}{K\sigma_1\sigma_2}\right] + \mu_{021}\left[-\frac{\rho_{23}-\rho_{12}\rho_{13}}{K\sigma_2\sigma_3}\right] + \mu_{030}\left[\frac{1-\rho_{13}^{2}}{K\sigma_2^{2}}\right]\right)$$

$$+ \frac{A_3}{2}\left(\mu_{120}\left[-\frac{\rho_{13}-\rho_{12}\rho_{23}}{K\sigma_1\sigma_3}\right] + \mu_{021}\left[\frac{1-\rho_{12}^{2}}{K\sigma_3^{2}}\right] + \mu_{030}\left[-\frac{\rho_{23}-\rho_{12}\rho_{13}}{K\sigma_2\sigma_3}\right]\right)$$

$$= w_1\frac{\sigma_2^{2}\sigma_3}{K\sigma_1}\rho_{23}^{2}(\rho_{12}\rho_{23}-\rho_{13}) + w_2\rho_{23}\sigma_2\sigma_3\left(1 - \frac{\rho_{23}}{K}(\rho_{23}-\rho_{12}\rho_{13})\right)$$

$$\frac{A_1}{2}\left(\mu_{102}\left[\frac{1-\rho_{23}^2}{K\sigma_1^2}\right] + \mu_{012}\left[-\frac{\rho_{12}-\rho_{13}\rho_{23}}{K\sigma_1\sigma_2}\right] + \mu_{003}\left[-\frac{\rho_{13}-\rho_{12}\rho_{23}}{K\sigma_1\sigma_3}\right]\right)$$

$$+\frac{A_2}{2}\left(\mu_{102}\left[-\frac{\rho_{12}-\rho_{13}\rho_{23}}{K\sigma_1\sigma_2}\right] + \mu_{012}\left[\frac{1-\rho_{13}^2}{K\sigma_2^2}\right] + \mu_{003}\left[-\frac{\rho_{23}-\rho_{12}\rho_{13}}{K\sigma_2\sigma_3}\right]\right) \qquad \text{(3.77 Contd)}$$

$$+\frac{A_3}{2}\left(\mu_{102}\left[-\frac{\rho_{13}-\rho_{12}\rho_{23}}{K\sigma_1\sigma_3}\right] + \mu_{012}\left[-\frac{\rho_{23}-\rho_{12}\rho_{13}}{K\sigma_2\sigma_3}\right] + \mu_{003}\left[\frac{1-\rho_{12}^2}{K\sigma_3^2}\right]\right)$$

$$= w_1\frac{\sigma_3^3}{K\sigma_1}(\rho_{12}\rho_{23}-\rho_{13}) + w_2\frac{\sigma_3^3}{K\sigma_2}(\rho_{12}\rho_{13}-\rho_{23})$$

For order 3 or greater equation 3.63 will suffice.

Thus one can see that "constraint" equations of order 0 and 1 are satisfied identically and that those of order 2 or more will simply provide some relationships between the higher order moments.

As stated before, the "constraint" equations can be solved and a solution for ψ_1 obtained. When the solution for ψ_1 is substituted in equation 3.40, the solution for f_v including the first order perturbation is

$$f_v = f_{vo}\left[1 + \sum_{i,j,k=0}^{\infty} A_{i,j,k}H_{i,j,k}(v_1,v_2,v_3)\right] \qquad (3.78)$$

where the $A_{i,j,k}$ coefficients satisfy the "constraint" equations.

Reconsider for a moment the general development of the theory. Since one has solved equation 3.39A for ψ_1, the next step in the procedure would be to replace ψ_1, in equation 3.39B and in a similar manner solve for ψ_2. Then one would proceed to solve for each ψ_i sequentially and finally would arrive at the complete solution for f_v. However, in the present work, equation 3.78 will be considered to be the solution for f_v.

CHAPTER IV

RELATIONSHIPS BETWEEN THEORY AND EXPERIMENT

Introduction

At the time that measurements of the turbulent velocity components in the atmosphere were made, simultaneous measurements of the spatially-dependent forces (F_i or $\frac{\partial \Omega}{\partial x_i}$) were not made. Indeed the force parameters could not be measured by the techniques that were used. Therefore the A_i coefficients (appearing in the differential equation for f_v and later appearing in the differential equations for ψ_i and the "constraint" equations) could not be obtained. Thus one could not proceed (with the data already taken) to solve the necessary equations and arrive at a solution for f_v in the manner given in Chapter III. In future atmospheric research, if techniques are developed for the measurement of the spatially-dependent parameters, the problems could be corrected by simultaneously measuring the spatially-dependent forces, the shear vector components, and the turbulent velocity components. In future laboratory research, the difficulties would not arise if one first measured the shear vector components and the forces F_i and then maintained similar flow characteristics while one measured the turbulent velocity components.

Next an attempt was made to get around these problems with the data already taken. The method involved measuring the third order moments (μ_{300},μ_{030},μ_{003},μ_{210},μ_{120},μ_{201},μ_{102},μ_{021},μ_{012}, and μ_{111}) and then using three of the second order "constraint" equations (equations 3.77) in an artificial sense as a set of simultaneous equations for A_1,A_2, and A_3.

Recall the discussion of variance formulas in Appendix A (equations A.55 through A.79), and recall that the standard error of a statistic is equal to the square root of the variance of the sampling distribution for that statistic. It is apparent from the formulas that variance of a statistic can become quite large both as the order of the statistic increases and as the multivariate character of the statistic appears. The above factors might have been tolerable, except for the fact that, in applying the "constraint" equations as a set of equations for the A_i, the errors involved in the "coefficients" of the A_i are not additive. Rather these errors must be obtained through the use of equations A.73 and A.74 which give the variances of functions of random variables (and indeed the "coefficients" of the A_i are complex functions of the random variables). Thus the method of getting around the problems with the previously taken data was dealt a fatal blow.

Despite the shortcomings in the data mentioned above, several illuminating relationships between the experimental data and the theory can be developed.

Truncation Relations

In the development of the solution for f_v which included the first-order perturbation ψ_1, an infinite series of trivariate Hermite polynomials was used. A practical representation of f_v requires truncation of this series at some acceptable level. In this work the series will be truncated at order 4. Note that this approximation is the lowest level of truncation which would include perturbation terms affecting both the even and odd order moments of f_v, and note that the "constraint" equations of order 5 or greater are all homogeneous equations so that the trivial solution for the A_{ijk} coefficients could be accepted. So truncation at order 4 requires

$$A_{i,j,k} = 0 \qquad \text{for} \qquad i + j + k \geq 5 \tag{4.1}$$

This truncation condition for the coefficients $A_{i,j,k}$ provides relations between moments of different orders. The relations between moments arise in the following manner. Recall equation 3.69 and consider the case where i=5, j=1, and k=0. Substitution from 4.1 into 3.69 then gives

$$0 = (5!1!0!)^{-1} \quad \langle G_{5,1,0} \rangle \tag{4.2}$$

If one substitutes for $G_{5,1,0}$ from equation F.100, one obtains

$$\begin{aligned} 0 = {} & \langle v_1{}^5 v_2 \rangle - 5\, \rho_{12}\sigma_1\sigma_2 \langle v_1{}^4 \rangle \\ & - 10\, \sigma_1{}^2 \langle v_1{}^3 v_2 \rangle + 30\, \rho_{12}\sigma_1{}^3\sigma_2 \langle v_1{}^2 \rangle \\ & + 15\, \sigma_1{}^4 \langle v_1 v_2 \rangle - 15\, \rho_{12}\sigma_1{}^5\sigma_2 \end{aligned} \tag{4.3}$$

Recall equations A.18, A.28, A.29, the fact that by definition the mean values of the turbulent velocity components are zero, and equation 3.9. The use of the above conditions gives, upon rearrangement

$$\mu_{5,1,0} = 5\,\mu_{400}\mu_{110} + 10\,\mu_{310}\mu_{200} - 30\,\mu_{200}^2\mu_{110} \tag{4.4}$$

In a similar manner one obtains the following relations between moments of order 5 and 6 and moments of lower order.

$$\mu_{500} = 10\,\mu_{300}\mu_{200}$$

$$\mu_{050} = 10\,\mu_{030}\mu_{020}$$

$$\mu_{005} = 10\,\mu_{003}\mu_{002}$$

$$\mu_{410} = 4\,\mu_{300}\mu_{110} + 6\,\mu_{210}\mu_{200}$$

$$\mu_{401} = 4\,\mu_{300}\mu_{101} + 6\,\mu_{201}\mu_{200} \tag{4.5}$$

$$\mu_{140} = 4\,\mu_{030}\mu_{110} + 6\,\mu_{120}\mu_{020}$$

$$\mu_{104} = 4\,\mu_{003}\mu_{101} + 6\,\mu_{102}\mu_{002}$$

$$\mu_{041} = 4\,\mu_{030}\mu_{011} + 6\,\mu_{021}\mu_{020}$$

$$\mu_{014} = 4\ \mu_{003}\mu_{011} + 6\ \mu_{012}\mu_{002}$$

$$\mu_{320} = \mu_{300}\mu_{020} + 3\ \mu_{120}\mu_{200} + 6\ \mu_{210}\mu_{110}$$

$$\mu_{230} = \mu_{030}\mu_{200} + 3\ \mu_{210}\mu_{020} + 6\ \mu_{120}\mu_{110}$$

$$\mu_{302} = \mu_{300}\mu_{002} + 3\ \mu_{102}\mu_{200} + 6\ \mu_{201}\mu_{101}$$

$$\mu_{203} = \mu_{003}\mu_{200} + 3\ \mu_{201}\mu_{002} + 6\ \mu_{102}\mu_{101}$$

$$\mu_{032} = \mu_{030}\mu_{002} + 3\ \mu_{012}\mu_{020} + 6\ \mu_{021}\mu_{011}$$

$$\mu_{023} = \mu_{003}\mu_{020} + 3\ \mu_{021}\mu_{002} + 6\ \mu_{012}\mu_{011} \qquad \text{(4.5 Cont.)}$$

$$\mu_{311} = \mu_{300}\mu_{011} + 3\ \mu_{201}\mu_{110} + 3\ \mu_{210}\mu_{101} + 3\ \mu_{111}\mu_{200}$$

$$\mu_{131} = \mu_{030}\mu_{101} + 3\ \mu_{021}\mu_{110} + 3\ \mu_{120}\mu_{011} + 3\ \mu_{111}\mu_{020}$$

$$\mu_{113} = \mu_{003}\mu_{110} + 3\ \mu_{012}\mu_{101} + 3\ \mu_{102}\mu_{011} + 3\ \mu_{111}\mu_{002}$$

$$\mu_{221} = 2\ \mu_{120}\mu_{101} + 2\ \mu_{210}\mu_{011} + \mu_{201}\mu_{020} + \mu_{021}\mu_{200} + 4\ \mu_{111}\mu_{110}$$

$$\mu_{212} = 2\ \mu_{102}\mu_{110} + 2\ \mu_{201}\mu_{011} + \mu_{210}\mu_{002} + \mu_{012}\mu_{200} + 4\ \mu_{111}\mu_{101}$$

$$\mu_{122} = 2\ \mu_{012}\mu_{110} + 2\ \mu_{021}\mu_{101} + \mu_{120}\mu_{002} + \mu_{102}\mu_{020} + 4\ \mu_{111}\mu_{011}$$

$$\mu_{600} = 15\ \mu_{400}\mu_{200} - 45\ \mu^3_{200} + 15\ \mu^3_{200} = 15\ \mu_{400}\mu_{200} - 30\ \mu^3_{200}$$

$$\mu_{060} = 15\ \mu_{040}\mu_{020} - 30\ \mu^3_{020}$$

$$\mu_{006} = 15\ \mu_{004}\mu_{002} - 30\ \mu^3_{002}$$

$$\mu_{510} = 5\ \mu_{400}\mu_{110} + 10\ \mu_{310}\mu_{200} - 30\ \mu^2_{200}\mu_{110}$$

$$\mu_{150} = 5\ \mu_{040}\mu_{110} + 10\ \mu_{130}\mu_{020} - 30\ \mu^2_{020}\mu_{110}$$

$$\mu_{501} = 5\ \mu_{400}\mu_{101} + 10\ \mu_{301}\mu_{200} - 30\ \mu^2_{200}\mu_{101}$$

$$\mu_{105} = 5\ \mu_{004}\mu_{101} + 10\ \mu_{103}\mu_{002} - 30\ \mu^2_{002}\mu_{101} \qquad \text{(4.5 Cont.)}$$

$$\mu_{051} = 5\ \mu_{040}\mu_{011} + 10\ \mu_{031}\mu_{020} - 30\ \mu^2_{020}\mu_{011}$$

$$\mu_{015} = 5\ \mu_{004}\mu_{011} + 10\ \mu_{013}\mu_{002} - 30\ \mu^2_{002}\mu_{011}$$

$$\mu_{420} = \mu_{400}\mu_{020} + 6\ \mu_{220}\mu_{200} + 8\ \mu_{310}\mu_{110} - 6\ \mu^2_{200}\mu_{020} - 24\ \mu_{200}\mu^2_{110}$$

$$\mu_{240} = \mu_{040}\mu_{200} + 6\ \mu_{220}\mu_{020} + 8\ \mu_{130}\mu_{110} - 6\ \mu_{200}\mu^2_{020} - 24\ \mu_{020}\mu^2_{110}$$

$$\mu_{402} = \mu_{400}\mu_{002} + 6\ \mu_{202}\mu_{200} + 8\ \mu_{301}\mu_{101} - 6\ \mu^2_{200}\mu_{002} - 24\ \mu_{200}\mu^2_{101}$$

$$\mu_{204} = \mu_{004}\mu_{200} + 6\ \mu_{202}\mu_{002} + 8\ \mu_{103}\mu_{101} - 6\ \mu_{200}\mu^2_{002} - 24\ \mu_{002}\mu^2_{101}$$

$$\mu_{042} = \mu_{040}\mu_{002} + 6\,\mu_{022}\mu_{020} + 8\,\mu_{031}\mu_{011} - 6\,\mu_{020}^2\mu_{002} - 24\,\mu_{020}\mu_{011}^2$$

$$\mu_{024} = \mu_{004}\mu_{020} + 6\,\mu_{022}\mu_{002} + 8\,\mu_{013}\mu_{011} - 6\,\mu_{020}\mu_{002}^2 - 24\,\mu_{002}\mu_{011}^2$$

$$\mu_{411} = \mu_{400}\mu_{011} + 4\,\mu_{301}\mu_{110} + 4\,\mu_{310}\mu_{101}$$

$$+ 6\,\mu_{211}\mu_{200} - 6\,\mu_{200}^2\mu_{011} - 24\,\mu_{200}\mu_{110}\mu_{101}$$

$$\mu_{141} = \mu_{040}\mu_{101} + 4\,\mu_{031}\mu_{110} + 4\,\mu_{130}\mu_{011}$$

$$+ 6\,\mu_{121}\mu_{020} - 6\,\mu_{020}^2\mu_{101} - 24\,\mu_{020}\mu_{110}\mu_{011}$$

$$\mu_{114} = \mu_{004}\mu_{110} + 4\,\mu_{013}\mu_{101} + 4\,\mu_{103}\mu_{011} \qquad \text{(4.5 Cont.)}$$

$$+ 6\,\mu_{002}\mu_{112} - 6\,\mu_{002}^2\mu_{110} - 24\,\mu_{002}\mu_{101}\mu_{011}$$

$$\mu_{330} = 3\,\mu_{130}\mu_{200} + 3\,\mu_{310}\mu_{020} + 9\,\mu_{220}\mu_{110}$$

$$- 18\,\mu_{200}\mu_{020}\mu_{110} - 12\,\mu_{110}^3$$

$$\mu_{303} = 3\,\mu_{103}\mu_{200} + 3\,\mu_{301}\mu_{002} + 9\,\mu_{202}\mu_{101}$$

$$- 18\,\mu_{200}\mu_{002}\mu_{101} - 12\,\mu_{101}^3$$

$$\mu_{033} = 3\,\mu_{013}\mu_{020} + 3\,\mu_{031}\mu_{002} + 9\,\mu_{022}\mu_{011}$$

$$- 18\ \mu_{020}\mu_{002}\mu_{011} - 12\ \mu_{011}^{3}$$

$$\mu_{321} = 3\ \mu_{220}\mu_{101} + 2\ \mu_{310}\mu_{011} + \mu_{301}\mu_{020} + 6\ \mu_{211}\mu_{110}$$

$$+ 3\ \mu_{121}\mu_{200} - 12\ \mu_{200}\mu_{110}\mu_{011} - 6\ \mu_{200}\mu_{020}\mu_{101} - 12\ \mu_{110}^{2}\mu_{101}$$

$$\mu_{231} = 3\ \mu_{220}\mu_{011} + 2\ \mu_{130}\mu_{101} + \mu_{031}\mu_{200} + 6\ \mu_{121}\mu_{110}$$

$$+ 3\ \mu_{211}\mu_{020} - 12\ \mu_{020}\mu_{110}\mu_{101} - 6\ \mu_{200}\mu_{020}\mu_{011} - 12\ \mu_{110}^{2}\mu_{011}$$

$$\mu_{312} = 3\ \mu_{202}\mu_{110} + 2\ \mu_{301}\mu_{011} + \mu_{310}\mu_{002} + 6\ \mu_{211}\mu_{101}$$

$$+ 3\ \mu_{112}\mu_{200} - 12\ \mu_{200}\mu_{101}\mu_{011} - 6\ \mu_{200}\mu_{002}\mu_{110} - 12\ \mu_{101}^{2}\mu_{110}$$

$$\mu_{213} = 3\ \mu_{202}\mu_{011} + 2\ \mu_{103}\mu_{110} + \mu_{013}\mu_{200} + 6\ \mu_{112}\mu_{101} \qquad \text{(4.5 Cont.)}$$

$$+ 3\ \mu_{211}\mu_{002} - 12\ \mu_{002}\mu_{101}\mu_{110} - 6\ \mu_{200}\mu_{002}\mu_{011} - 12\ \mu_{101}^{2}\mu_{011}$$

$$\mu_{132} = 3\ \mu_{022}\mu_{110} + 2\ \mu_{031}\mu_{101} + \mu_{130}\mu_{002} + 6\ \mu_{121}\mu_{011}$$

$$+ 3\ \mu_{112}\mu_{020} - 12\ \mu_{020}\mu_{101}\mu_{011} - 6\ \mu_{020}\mu_{002}\mu_{110} - 12\ \mu_{011}^{2}\mu_{110}$$

$$\mu_{123} = 3\ \mu_{022}\mu_{101} + 2\ \mu_{013}\mu_{110} + \mu_{103}\mu_{020} + 6\ \mu_{112}\mu_{011}$$

$$+ 3\ \mu_{121}\mu_{002} - 12\ \mu_{002}\mu_{011}\mu_{110} - 6\ \mu_{020}\mu_{002}\mu_{101} - 12\ \mu_{011}^{2}\mu_{101}$$

$$\mu_{222} = 4\,\mu_{112}\mu_{110} + 4\,\mu_{121}\mu_{101} + 4\,\mu_{211}\mu_{011} + \mu_{220}\mu_{002}$$

$$+ \mu_{022}\mu_{200} + \mu_{202}\mu_{020} - 2\,\mu_{200}\mu_{020}\mu_{002} \qquad \text{(4.5 Cont.)}$$

$$- 4\,\mu_{002}\mu_{110}^2 - 4\,\mu_{020}\mu_{101}^2 - 4\,\mu_{200}\mu_{011}^2$$

$$- 16\,\mu_{110}\mu_{101}\mu_{011}$$

It should again be noted that the above relations between moments of different orders arise from truncation conditions and not from basic considerations in Chapter III. In a procedure analogous to that performed by Frenkiel and Klebanoff (1967) (who used the method to compare their ad hoc, two-dimensional, non-Gaussian p.d.f. with their experimental data), the truncation relation 4.5 can be used for comparison of the approximation with the experimental data.

The procedure is simple. Consider the equation 4.4; in the first case one would experimentally compute the moment μ_{510} from the observed p.d.f., whereas in the second case one would experimentally measure the moments μ_{400}, μ_{100}, μ_{310}, and μ_{200} then one would compute a value for μ_{510} using the truncation relation 4.4. In order to non-dimensionalize and scale for comparison with the Gaussian predicted value, each value for μ_{510} should be divided by $(\sigma_1{}^5\sigma_2\rho_{12})$. The procedure for other truncation relations would be similar.

The results of the application of the procedure to several of the truncation relations are shown in Figures 22, 23, 24, and 25. In all four figures the direct computation of the value for the moment

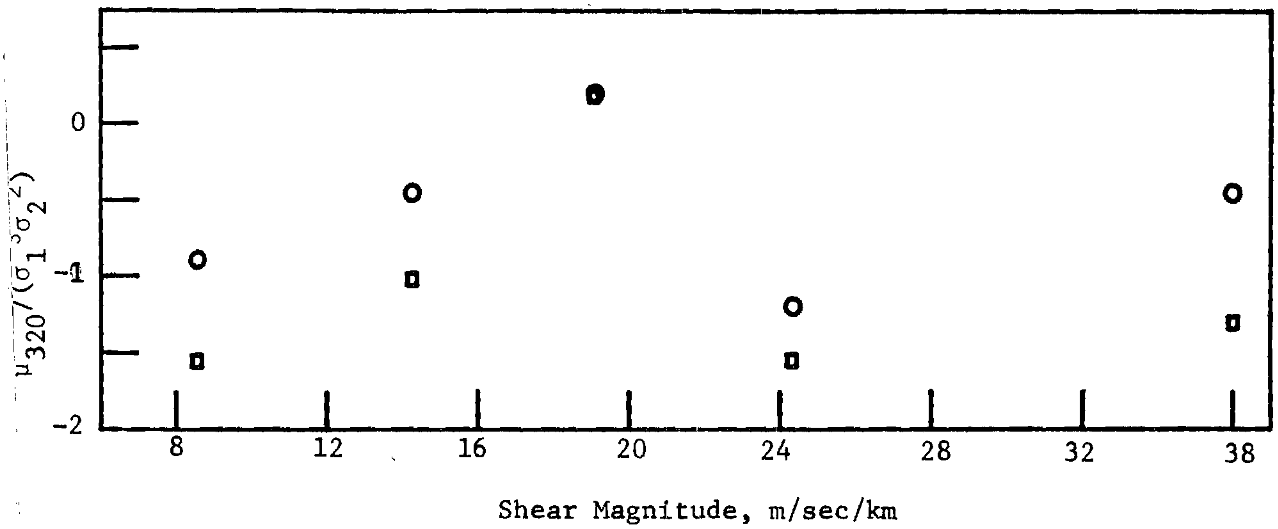

Figure 22. Experimental and Truncation Values for $\mu_{320}/(\sigma_1{}^3\sigma_2{}^2)$.
(O indicates truncation value, and ◻ indicates experimental value.)

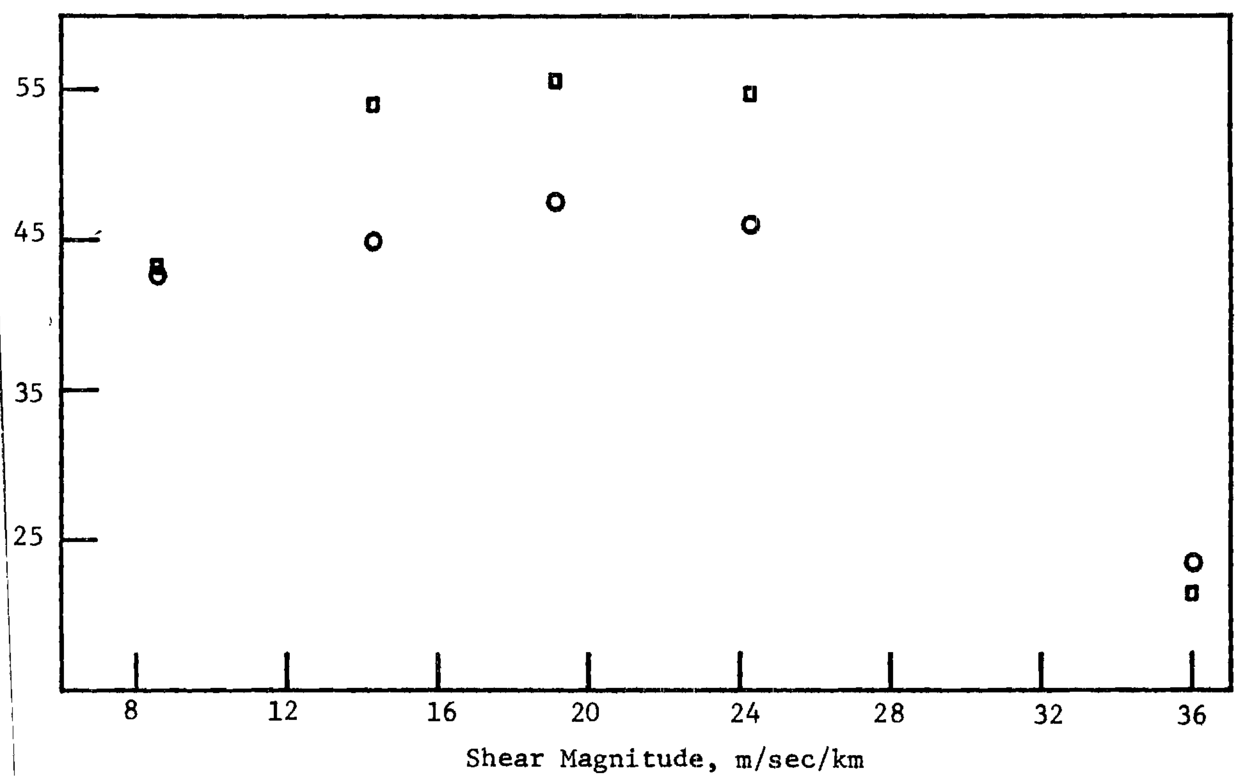

Figure 23. Experimental and Truncation Values for $\mu_{600}/\sigma_1{}^6$.
(O indicates truncation value, and ◻ indicates experimental value.)

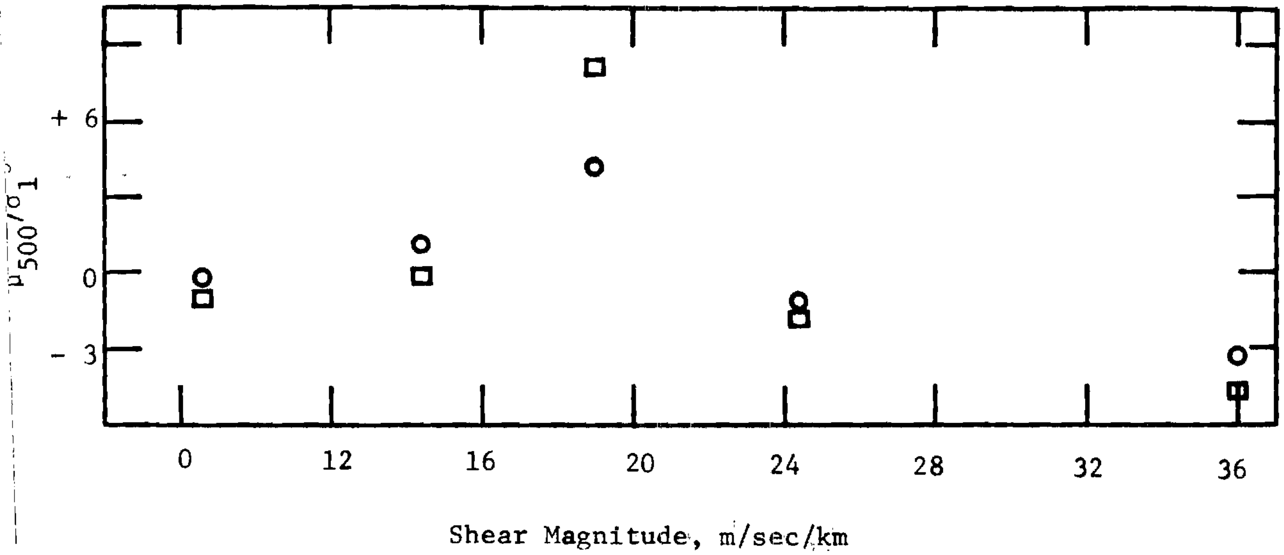

Figure 24. Experimental and Truncation values for μ_{500}/σ_1^5. (O indicates truncation value, and □ indicates experimental value.)

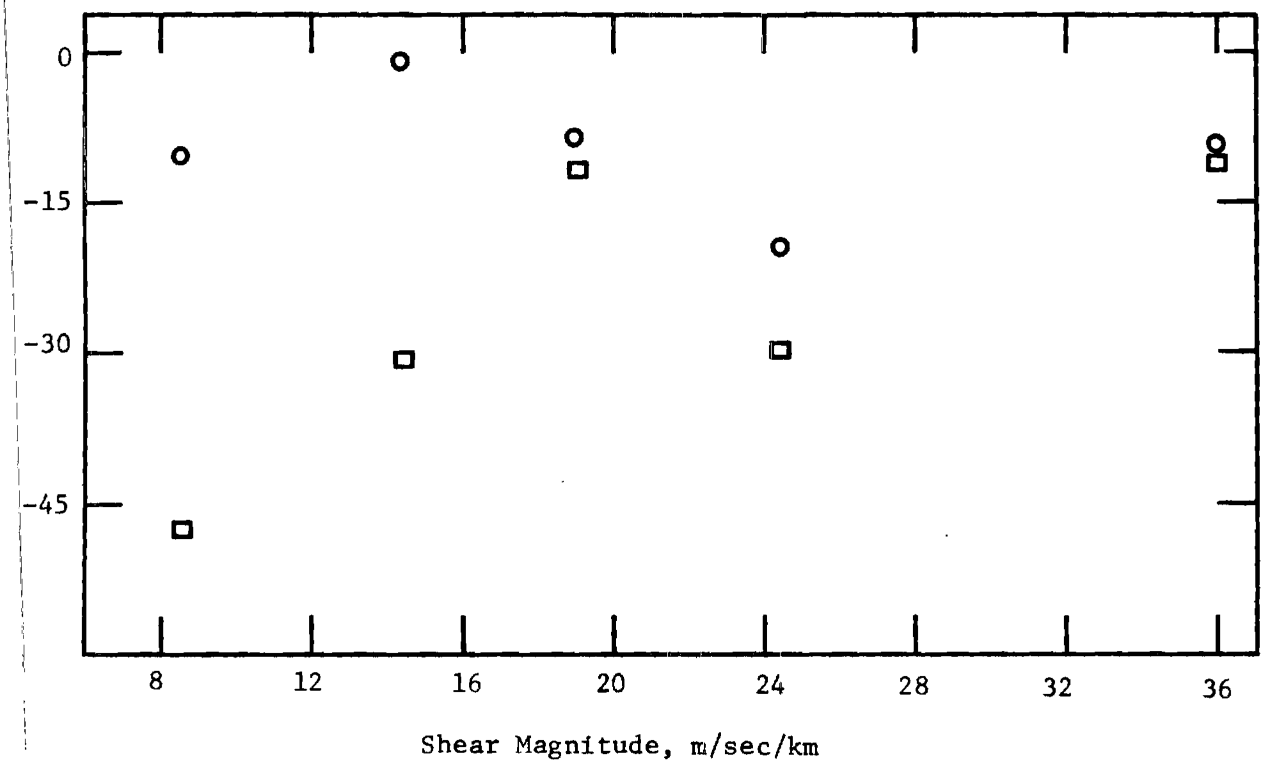

Figure 25. Experimental and Truncation values for $\mu_{401}/(\sigma_1^4\sigma_3\rho_{13})$. (O indicates truncation value, and □ indicates experimental value.)

from the experimental data is indicated by the symbol □ , whereas the computation of the value for the moment using the truncation relation is indicated by the symbol ○ . In order to obtain more than one pair of values for a given moment, the data were divided into the five shear magnitude intervals.

Figure 22 shows the pairs of values of $(\mu_{320}/[\sigma_1{}^3\sigma_2{}^2])$ plotted versus the average value of the shear magnitude in each interval. It can be seen that the truncation values compare rather well with the experimental values, particularly since a Gaussian p.d.f. would give the constant value of zero for the moment regardless of the shear value.

Figure 23 illustrates the pairs of values of $(\mu_{600}/\sigma_1{}^6)$ plotted versus shear magnitude. The truncation values provide a far better estimate of the experimental values than the constant value of fifteen which would be predicted by a Gaussian p.d.f. Figures 24 and 25 present the results of the computation of pairs of values for the moments $(\mu_{500}/\sigma_1{}^5)$ and $(\mu_{401}/[\sigma_1{}^4\sigma_3\rho_{13}])$ respectively. Here again, as is shown in both figures, the truncation values follow the variations in the experimental values from one shear interval to the next, and the truncation values provide better estimates of the experimental values than estimates based on a Gaussian p.d.f. which are constant at the value zero for both moments.

Pairs of values for other moments were computed and the values computed from the truncation relation provided better agreement with the experimental values than values which would have been predicted by a Gaussian p.d.f.

Computation of One-Dimensional Probability Density Functions

The probability density functions that were plotted in Chapter II were the p.d.f. for the marginal distributions (one-dimensional p.d.f.'s for the univariate distributions, see Appendix A). Recall that the solution for f_v which included the first order-perturbation is

$$f_v(v_1,v_2,v_3) = f_{vo}(v_1,v_2,v_3)[1 + \sum_{i,j,k=0}{}' A_{i,j,k}H_{i,j,k}(v_1,v_2,v_3)]$$

where $\sum'$ indicates that the summation has been truncated at order 4 in the trivariate Hermite polynomials. Then from equations A.11, A.12, and A.13, one can see that the one-dimensional p.d.f.'s $f_1(v_1)$, $f_2(v_2)$, and $f_3(v_3)$ are given by

$$f_1(v_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_v(v_1,v_2,v_3)\; dv_2 dv_3 \tag{4.6}$$

$$f_2(v_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_v(v_1,v_2,v_3)\; dv_1 dv_3 \tag{4.7}$$

$$f_3(v_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_v(v_1,v_2,v_3)\; dv_1 dv_2 \tag{4.8}$$

where $f_1(v_1)$ is the p.d.f. for the parallel component of the turbulent velocity, $f_2(v_2)$ is the p.d.f. for the perpendicular component of the turbulent velocity, and $f_3(v_3)$ is the p.d.f. for the vertical component of the turbulent velocity.

Consider the integral

$$\int\int_{-\infty}^{\infty} H_{j,k,\ell}[(2\pi)^3 K \sigma_1^2\sigma_2^2\sigma_3^2]^{-1/2} \exp\{-\frac{1}{2K}[\frac{(1-\rho_{23}^2)}{\sigma_1^2} v_1^2$$

$$-\frac{2(\rho_{12}-\rho_{13}\rho_{23})}{\sigma_1\sigma_2} v_1v_2 - \frac{2(\rho_{13}-\rho_{12}\rho_{23})}{\sigma_1\sigma_3} v_1v_3 + \frac{(1-\rho_{13}^2)}{\sigma_2^2} v_2^2$$

$$-\frac{2(\rho_{23}-\rho_{12}\rho_{13})}{\sigma_2\sigma_3} v_2v_3 + \frac{(1-\rho_{12}^2)}{\sigma_3^2} v_3^2]\} dv_2dv_3 \tag{4.9}$$

where $k+j+\ell \leq 4$, and note that all of the integrals of f_v to obtain $f_1(v_1)$ will be of this type. Let

$$p = \frac{(1-\rho_{12}^2)}{2 K\sigma_3^2} \tag{4.10}$$

$$q = \frac{(\rho_{23}-\rho_{12}\rho_{13})v_2}{2 K \sigma_2\sigma_3} + \frac{(\rho_{13}-\rho_{12}\rho_{23})v_1}{2 K \sigma_1\sigma_3} \tag{4.11}$$

From equation C.4, it can be seen that the result of the integration with respect to the variable v_3 is

$$\int_{-\infty}^{\infty} [(2\pi)^2 \sigma_1^2\sigma_2^2)]^{-1/2} L(v_1,v_2) \exp\{-\frac{v_1^2}{2 \sigma_1^2(1-\rho_{12}^2)}\}[1-\rho_{12}^2]^{-1/2}$$

$$\cdot\exp\left\{-\frac{1}{2(1-\rho_{12}^2)}\left[\frac{v_2^2}{\sigma_2^2}-\frac{2\rho_{12}v_1v_2}{\sigma_1\sigma_2}\right]\right\}\ dv_2 \tag{4.12}$$

where $L(v_1,v_2)$ is a polynomial in v_1 and v_2 of order $n=j+k+\ell$. By a polynomial in v_1 and v_2 of order n, it is meant that the highest power of a term in v_1 or v_2 alone is n and that the sum of the exponents of a term which contains a product of v_1 and v_2 must be less than or equal to n. Now let

$$p = \frac{1}{2(1-\rho_{12}^2)\sigma_2^2} \tag{4.13}$$

$$q = \frac{\rho_{12}v_1}{2(1-\rho_{12}^2)\sigma_1\sigma_2} \tag{4.14}$$

again from equation C.4 it can be seen that the result of integration with respect to the variable v_2 is

$$[2\pi\sigma_1^2]^{-1/2}\exp\left\{-\frac{v_1^2}{2\sigma_1^2}\right\} M(v_1) = f_{10}M(v_1) \tag{4.15}$$

where $M(v_1)$ is a polynomial of order n, and where f_{10} is defined by

$$f_{10} = [2\pi\sigma_1^2]^{-1/2}\exp\left\{-\frac{v_1^2}{2\sigma_1^2}\right\} \tag{4.16}$$

Since the series of trivariate Hermite polynomials was truncated at order four, all of the integrals of f_v to obtain $f_1(v_1)$ will give results of the form 4.15 where the polynomials are of order four or less. These polynomials could be combined into one polynomial, so one has

$$f_1(v_1) = f_{10}[N(v_1)] \tag{4.17}$$

where $N(v_1)$ is a polynomial of order 4 in v_1. This polynomial $N(v_1)$ can be represented by a sum of the first five univariate Hermite polynomials

$$f_1(v_1) = f_{10}[D_0H_0 + D_1H_1(v_1) + D_2H_2(v_1) + D_3H_3(v_1) + D_4H_4(v_1)] \tag{4.18}$$

In order to introduce the proper dimensionality into the non-dimensional Hermite polynomials given in equation F.13, one simply starts with the derivative definition

$$H_n(v_1) = (-1)^n \exp\{\frac{v_1^{\,2}}{2\sigma_1^2}\} \frac{d^n}{dv_1^{\,n}} \exp\{-\frac{v_1^{\,2}}{2\,\sigma_1^2}\} \tag{4.19}$$

Thus one obtains

$$
\begin{aligned}
H_0(v_1) &= 1 && = G_{0,0,0}(v_1,v_2,v_3) \\
H_1(v_1) &= v_1/\sigma_1^2 && = G_{1,0,0}(v_1,v_2,v_3)/\sigma_1^2 \qquad (4.20) \\
H_2(v_1) &= \frac{v_1^{\,2}}{\sigma_1^{\,4}} - \frac{1}{\sigma_1^{\,2}} && = G_{2,0,0}(v_1,v_2,v_3)/\sigma_1^4
\end{aligned}
$$

$$H_3(v_1) = \frac{v_1^3}{\sigma_1^6} - \frac{3v_1}{\sigma_1^4} \qquad = G_{3,0,0}(v_1,v_2,v_3)/\sigma_1^6$$

(4.20 Cont.)

$$H_4(v_1) = \frac{v_1^4}{\sigma_1^8} - 6\frac{v_1^2}{\sigma_1^6} + \frac{3}{\sigma_1^4} \qquad = G_{4,0,0}(v_1,v_2,v_3)/\sigma_1^8$$

$$\text{or } H_i(v_1) = G_{i,0,0}(v_1,v_2,v_3)/\sigma_1^{2i} = \frac{H_i(x)}{\sigma_1^i} \tag{4.21}$$

where $x = v_1/\sigma_1$. So one has

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_v dv_2 dv_3 = f_{10}(D_0H_0 + D_1H_1 + D_2H_2 + D_3H_3 + D_4H_4) \tag{4.22}$$

In order to determine the coefficient D_i, multiply both sides by H_i $(i \leq 4)$ and integrate with respect to v_1

$$\int_{-\infty}^{\infty} \frac{G_{i,0,0}}{\sigma_1^{2i}} \left\{ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_v dv_2 dv_3 \right\} dv_1 = \int_{-a}^{\infty} H_i f_{10}(D_0H_0 + D_1H_1 + D_2H_2$$

$$+ D_3H_3 + D_4H_4)\ dv_1 \tag{4.23}$$

The integral on the right hand side, by application of F.13, becomes

$$\int_{-\infty}^{\infty} H_i f_{10}(D_0H_0 + D_1H_1 + D_2H_2 + D_3H_3 + D_4H_4)\ dv_1 = D_i\frac{i!}{\sigma_1^{2i}} \text{ for } i \leq 4$$

(4.24)

Since the order of integration may be interchanged, the left hand integral, by equation F.73, becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{G_{i00}}{\sigma_1^{2i}} f \, dv_1 dv_2 dv_3 = A_{i,0,0} \frac{i!}{\sigma_1^{2i}} \quad \text{where } i \leq 4 \tag{4.25}$$

Therefore

$$D_i = A_{i,0,0} \tag{4.26}$$

and

$$f_1(v_1) = f_{10}(v_1) \left[\sum_{i=0}^{4} A_{i,0,0} G_{i,0,0} / \sigma_1^{2i} \right] \tag{4.27}$$

In an exactly analogous manner, one obtains

$$f_2(v_2) = f_{20}(v_2) \left[\sum_{i=0}^{4} A_{0,i,0} G_{0,i,0} / \sigma_2^{2i} \right] \tag{4.28}$$

and

$$f_3(v_3) = f_{30}(v_3) \left[\sum_{i=0}^{4} A_{0,0,i} G_{0,0,i} / \sigma_3^{2i} \right] \tag{4.29}$$

where

$$f_{20}(v_2) = [2\pi\sigma_2^{2}]^{-1/2} \exp\left\{- \frac{v_2^{2}}{2\sigma_2^{2}}\right\} \tag{4.30}$$

and

$$f_{30}(v_3) = [2\pi\sigma_3^{\,2}]^{-1/2} \quad \exp\{-\frac{v_3^{\,2}}{2\,\sigma_3^{\,2}}\} \tag{4.31}$$

Equations 4.27, 4.28, and 4.29 give the one-dimensional p.d.f.'s for the parallel, perpendicular, and vertical components respectively of the turbulent velocity. If equation 4.27 is written out completely with substitutions from equations 3.73 and F.100, one has

$$\begin{aligned} f_1(v_1) \;=\; & [2\pi\mu_{200}]^{-1/2} \quad \exp\{-\frac{v_1^{\,2}}{2\,\mu_{200}}\} \cdot [1 \\ & + (6\mu_{200}^3)^{-1}\,\mu_{300}(v_1^{\,3} - 3\,\mu_{200}\,v_1) \\ & + (24\,\mu_{200}^4)^{-1}\,(\mu_{400} - 3\,\mu_{200}^2)(v_1^{\,4} - 6\,\mu_{200}v_1^{\,2} + 3\,\mu_{200}^2)] \end{aligned} \tag{4.32}$$

Now if one measures the moments μ'_{100}, μ_{200}, μ_{300}, and μ_{400} from the experimental data, then by plotting equation 4.32 versus velocity one can obtain a comparison between the experimental data and the theoretical solution for $f_1(v_1)$ including the first-order perturbation. Note that the moment μ'_{100} merely corrects for the inaccuracies which give a non-zero mean for the observed data.

Figure 26 provides a plot of $f_1(v_1)$ versus the parallel component of the velocity for all data in the shear system. The solid curve is $f_1(v_1)$. The dots give the experimentally observed p.d.f., and for further comparison the dashed curve gives a Gaussian p.d.f. with the appropriate mean and variance. It is immediately obvious from

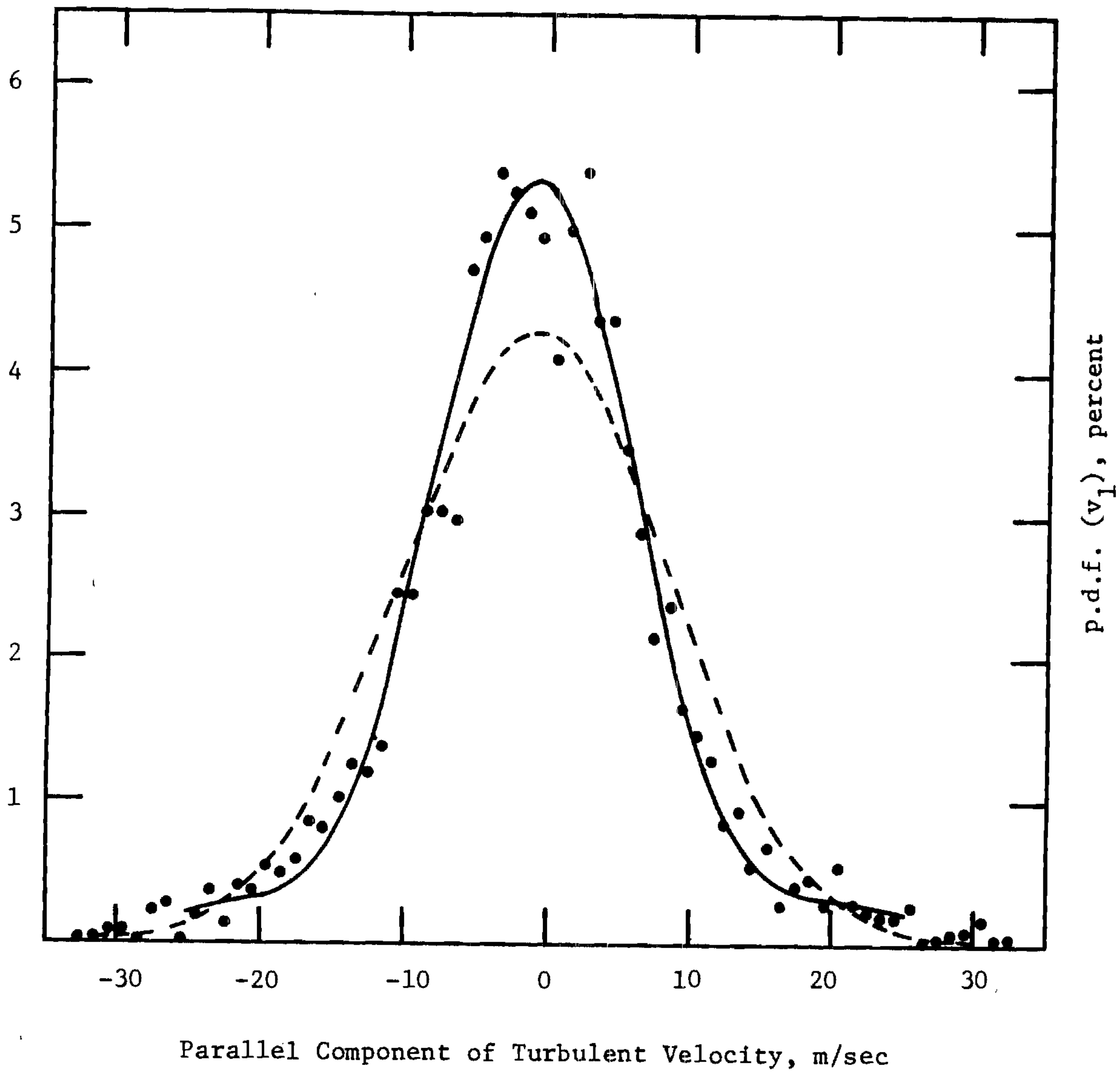

Figure 26. The p.d.f. for the Parallel Component of the Turbulent Velocity for all Data in the Shear System. (The dashed curve is a Gaussian distribution with the appropriate mean and standard deviation. The solid curve is the p.d.f. $f_1(v_1)$ where the necessary moments are the same as the observed distribution.)

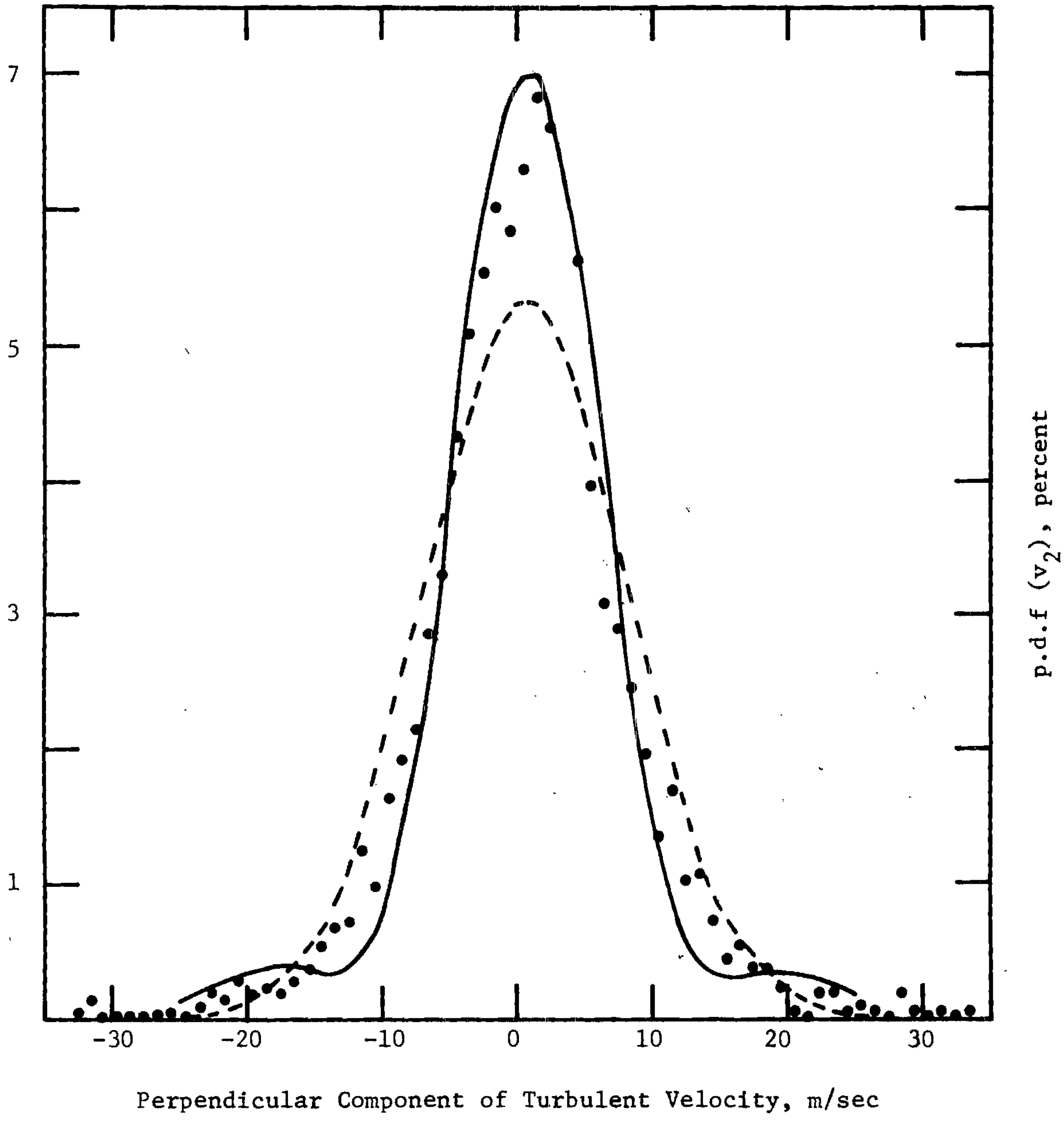

Figure 27. The p.d.f. for the Perpendicular Component of the Turbulent Velocity for all Data in the Shear System. (The dashed curve is a Gaussian distribution with the appropriate mean and standard deviation. The solid curve is the p.d.f. $f_2(v_2)$ where the necessary moments are the same as the observed distribution.)

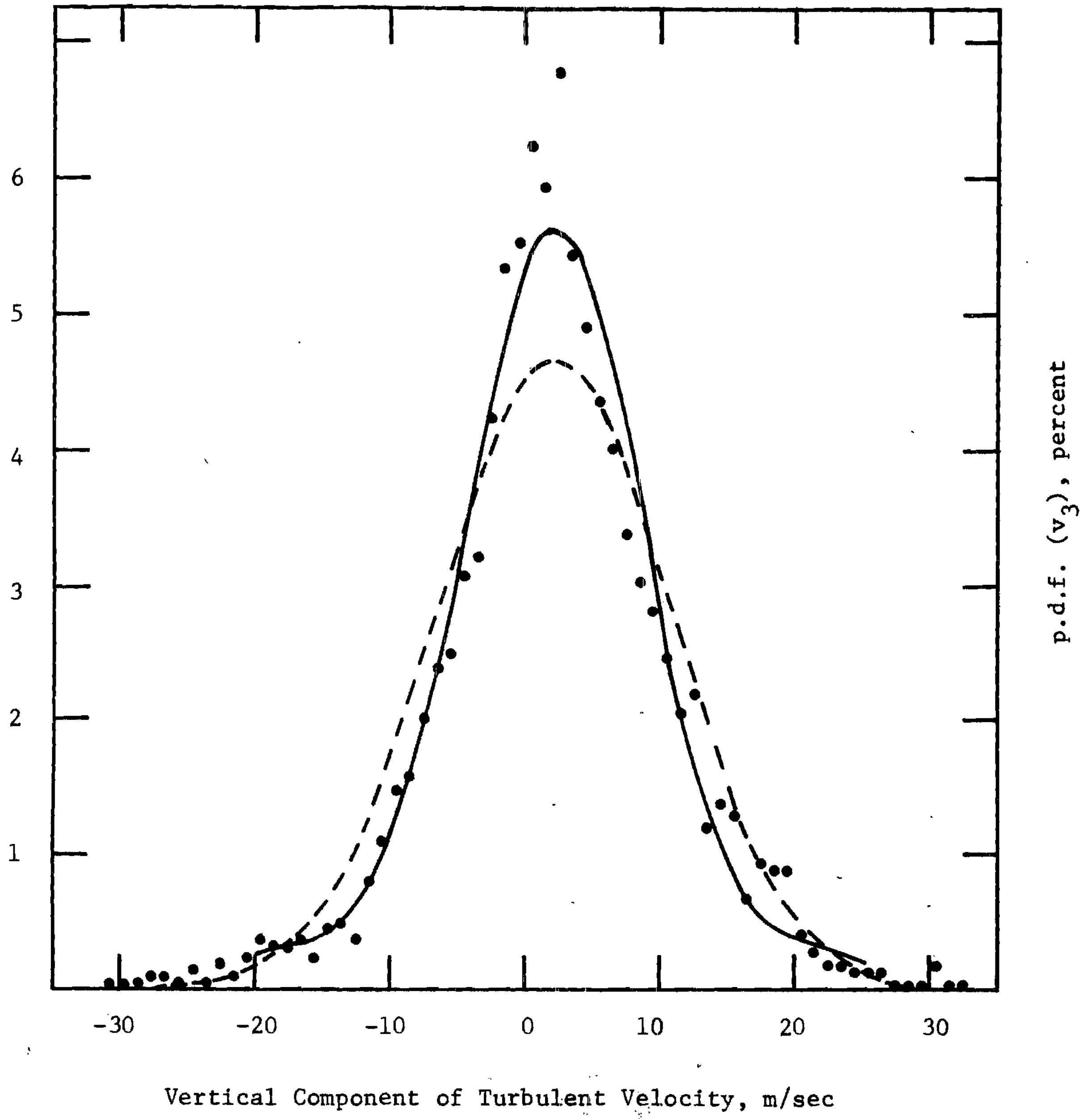

Figure 28. The p.d.f. for the Vertical Component of the Turbulent Velocity for all Data in the Shear System. (The dashed curve is a Gaussian distribution with the appropriate mean and standard deviation. The solid curve is the p.d.f. $f_3(v_3)$ where the necessary moments are the same as the observed distribution.)

the figure that $f_1(v_1)$ fits the experimental data far better than does the Gaussian p.d.f. The computed $f_1(v_1)$ has a high central peak and broad tails similar to the experimental data.

Figures 27 and 28 show plots of $f_2(v_2)$ and $f_3(v_3)$ (the solid curves) respectively for the perpendicular and vertical components of the velocity for all data in the shear system. In both figures the experimental data is represented by dots and the appropriate Gaussian p.d.f. is represented by the dashed curve. Again in each figure the computed $f_2(v_2)$ (in the first) and $f_3(v_3)$ (in the second) provide better agreement with the experimental data then does the Gaussian p.d.f. Both $f_2(v_2)$ and $f_3(v_3)$ exhibit the proper non-Gaussian characteristics.

CHAPTER V

CONCLUSIONS

The probability density functions of the turbulent velocity components were first computed in the geographic coordinate system in the 90 to 110 km region and were found to be non-Gaussian. The predominant non-Gaussian characteristic of the p.d.f.'s was a high flatness factor, β_2. The values of β_2 are significantly different at the 1% level from the value for a Gaussian p.d.f. These high values of β_2 could not have arisen through the simultaneous sampling of Gaussian errors and Gaussian turbulent velocities. It is highly improbable (at the 1% level) that the non-Gaussian character is due to an intermittent observation in the first case of Gaussian turbulent velocities plus Gaussian errors or in the alternative case of Gaussian errors alone. Thus the non-Gaussian characteristics are attributed to the turbulent velocities.

The observations were converted to the shear system, and β_2 was consistently non-Gaussian at the 1% level throughout the various break-ups of the data. The skewness, γ_1, becomes significantly different from zero at values of the shear magnitude > 17 m/sec/km. Further confirmation of the non-Gaussian character of the p.d.f.'s was obtained through the use of χ^2 tests. An analysis of variance through M-tests indicates that the shear vector definitely influences the turbulent velocity and thus contributes to the non-Gaussian character of the p.d.f. M-tests also show that an unspecified quantity which varied with altitude was influencing the turbulent velocities, and that over the lifetime of the releases the p.d.f. was relatively stationary.

Use of the above results in combination with a development similar to that found in classical statistical mechanics leads to a differential equation for the velocity-dependent probability density function f_v. The differential equation for f_v can be solved through a perturbation method of solution. The unperturbed solution is a trivariate Gaussian p.d.f., and the solution to the first-order perturbation differential equation is a series of trivariate Hermite polynomials. The differential equation places "constraint" equations on the A_{ijk} coefficients of the Hermite polynomials. These "constraint" equations provide, in actuality, relations between the moments of the distribution.

The solution f_v which includes the first-order perturbation solution gives substantially better agreement with the observed data than does a Gaussian p.d.f. This improved agreement is demonstrated through the truncation relations which provide relations between moments of different order and through the computation of the one-dimensional p.d.f.'s.

Thus the truncated solution for f_v (based upon the fact that the shear vector affects the turbulent velocity p.d.f.) accounts rather well for the non-Gaussian characteristics of the observed turbulent velocity probability density functions.

Suggestions for Future Research

As is the case with most research, the completion of this research endeavor provides several new directions along which further research could be initiated. Perhaps the most interesting would be the study of turbulent shear flow in a pipe. Through the use of a laser velocimeter, turbulent velocity components in all three directions could be measured simultaneously with no disturbance of the flow within the pipe due to the presence of the measuring device. Prior to the measurement of the turbulent velocities and under similar flow conditions, the spatially-dependent forces could be measured. This would allow one to solve directly the "constraint" equations for the A_{ijk} coefficients of the Hermite polynomials. The controllable nature of the experiment would allow investigation of the influences of the spatially-dependent parameters as well as a more detailed analysis of the variations of the p.d.f. with respect to shear magnitude.

Another research endeavor could involve the solution of the higher order perturbation equations. The solution of the first perturbation equation (which directly affects the third-order moments) would be known and the second-order perturbation equation (which, it is speculated, would affect the fourth and sixth-order moments) could be solved. Thus sequential solution of the perturbation equations would result in the complete solution for f_v. This procedure would be feasible if at each order of perturbation the homogeneous "constraint" equations similar to 3.63 are assumed to have only the trivial solutions.

APPENDIX A

STATISTICS

This appendix contains those definitions, formulas and statistical tests which are fundamental to the development of the present work. The derivations of the formulas and more detailed discussions of the definitions and tests can be found in the texts by the following authors: KEEPING [1962], KENDALL [1946, 1958], and WEATHERBURN [1961].

Definitions

variate: If the entire space of all possible events U is partitioned into a set of disjoint events A_1, A_2, A_3, . . . A_n, . . ., and if a function X is defined on all points of U by the relation $X(A_i) = x_i$ where $x_1, x_2, x_3, \ldots x_n, \ldots$ are real numbers, then X is called a simple random variable or a variate. Thus, the variate associates with each possible event A_i a real number x_i which may be the natural number connected with the event (for example the weight of a child at birth) or a more or less arbitrary number (for example when a male birth is denoted by one and a female birth by zero).

Probability distributions: If corresponding to the n exhaustive and mutually exclusive cases that may result from a sample (a set of experimental observations), a variate X assumes the n values x_i with corresponding probabilities $f(x_i)$, then the assemblage of values x_i, with their probabilities f_i, constitutes the probability distribution of the variate for that sample.

probability density function: If U cannot be partitioned into a finite (or even a denumerable) set of regions A_i, in each of which X takes a definite value x_i, then X varies continuously over some interval and there is a probability $f(x)dx$ that it takes a value between x and x + dx. Here f(x) is called a probability density function. It is a single-valued, non-negative function, integrable over its whole domain of definition, and such that

$$\int_{\text{Domain}} f(x)dx = 1 \ . \qquad (A.1)$$

Now if the event A corresponds to a value of x between a and b, the probability of A is defined by

$$\Pr(A) = \int_a^b f(x)\,dx \ . \qquad (A.2)$$

cumulative distribution function : For a continuous variate the probability that $X \leq x$ is given by

$$F(x) = \int_{-\infty}^{x} f(u)\, du \qquad (A.3)$$

The function, $F(x)$, is called the cumulative distribution function or simply the distribution function. dF will often be called the distribution.

For a discrete variate X which takes the distinct values x_i ($i = 1,2,3. . .$) with probabilities $f(x_i)$, the distribution function is defined as

$$F(x) = \sum_{x_i \leq x} f(x_i) \qquad (A.4)$$

expectation: For a discrete variate, if $f(x_j)$ is the probability of x_j, the expectation of the variate X is

$$E(X) = \sum_{j} x_j\, f(x_j) \qquad (A.5)$$

Now if $\psi(X)$ is a function of the variate X and if the function has the value $\psi(x_i)$ when X takes the value x_i and since $f(x_i)$ is the probability of this value of the function then the expected value of the function is

$$E[\psi(X)] = \sum_{i} \psi(x_i)\, f(x_i) \qquad (A.6)$$

For a continuous variate, if f(x) is the probability density function, the expectation of the variate X is

$$E(X) = \int_{\substack{\text{Domain}\\ \text{of}\\ \text{Definition}}} x f(x) dx \qquad (A.7)$$

Similarly, the generalization of A.6 to the continuous variate case gives

$$E[\psi(X)] = \int_{\text{Domain}} \psi(x)\ f\ (x)\ dx \qquad (A.8)$$

where ψ is a function of X and f is the probability density function.

Strictly speaking, different symbols should be used to denote the variate and the range of values which the variate can take, but in the present work the value will be the natural number associated with the event. Thus it will be convenient to denote both the variate and the range of values by the same symbol. Where this could lead to confusion, the stricter notation will be used.

trivariate probability density function: If there exist three variates (X_1,X_2, and X_3) which vary continuously over some volume and there is a probability $f(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3$ that the three variates simultaneously take values between x_1 and $x_1 + dx_1$, x_2 and x_2+dx_2,x_3 and $x_3 + dx_3$ respectively, then $f(x_1, x_2, x_3)$ is the trivariate probability density function. The expression probability density function will be abbreviated p.d.f., and the term cumulative distribution function will be abbreviated to c.d.f.

trivariate cumulative distribution function: The trivariate distribution with variates x_1, x_2, x_3 may be written as

$$dF = f(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3 \qquad \text{(A.9)}$$

where f is the trivariate p.d.f. Thus the trivariate cumulative distribution function is

$$F(x_1,x_2,x_3) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{x_3} f(u_1,u_2,u_3)\, du_1 du_2 du_3 \qquad \text{(A.10)}$$

marginal distributions: For any value of x_1, the total probability density is obtained by summing over all values of x_2 and x_3.

Then, if the c.d.f. is $F(x_1,x_2,x_3)$, the univariate c.d.f. of x_1 may be expressed as $F(x_1,\infty,\infty)$ with similar expressions for the univariate cumulative distribution functions of the other two variates. Thus the univariate distributions are

$$dF_1(x_1) = \left\{ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1,x_2,x_3)\ dx_2 dx_3 \right\} dx_1 \qquad (A.11)$$

$$dF_2(x_2) = \left\{ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1,x_2,x_3)\ dx_1 dx_3 \right\} dx_2 \qquad (A.12)$$

$$dF_3(x_3) = \left\{ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1,x_2,x_3)\ dx_1 dx_2 \right\} dx_3 \qquad (A.13)$$

These univariate distributions are called marginal distributions.

independent variates:

If conditions exist such that

$$F(x_1,x_2,x_3) = F(x_1,\infty,\infty)\ F(\infty,x_2,\infty)\ F(\infty,\infty,x_3)$$

$$= F_1(x_1)\ F_2(x_2)\ F_3(x_3) \qquad (A.14)$$

then the variates x_1, x_2, and x_3 are independent. Expressed in terms of the p.d.f., the above equation becomes

$$\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{x_3} f(x_1,x_2,x_3)\ dx_1dx_2dx_3$$

$$= \int_{-\infty}^{x_1} f_1(x_1)\ dx_1 \int_{-\infty}^{x_2} f_2(x_2)dx_2$$

$$\int_{-\infty}^{x_3} f_3(x_3)dx_3 \qquad \text{(A.15)}$$

and thus

$$f(x_1,x_2,x_3) = f_1(x_1)f_2(x_2)f_3(x_3) \qquad \text{(A.16)}$$

Therefore, when the c.d.f. and the p.d.f. factor into three parts, each solely dependent upon a different variate, the variates are statistically independent.

a statistic: The singular form "statistic" is defined as a function of the observations in a sample from a population.

moments: Trivariate moments of the parent population about the arbitrary points a_1 for v_1, a_2 for v_2, and a_3 for v_3 are defined by

$$\mu'_{qrs} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (v_1-a_1)^q\ (v_2-a_2)^r\ (v_3-a_3)^s dF \qquad \text{(A.17)}$$

where v_1, v_2, and v_3 are the three variates and dF is the trivariate distribution function. If two of the subscripts (q,r,s) are zero, the moment becomes the ordinary univariate moment of the marginal distribution; if one of the subscripts is zero, the moment becomes the bivariate moment. Now if we take a_1, a_2, and a_3 to be the coordinates of the origin, then μ'_{100} is the mean value for v_1, μ'_{010} is the mean value for v_2, and μ'_{001} is the mean value for v_3. Therefore the moments about the mean are defined by

$$\mu_{qrs} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (v_1-\mu'_{100})^q \, (v_2-\mu'_{010})^r \, (v_3 - \mu'_{001})^s \, dF \qquad (A.18)$$

Trivariate moments of the sample are defined so that

$$m'_{qrs} = \frac{1}{N} \sum_{i=1}^{N} (v_{1_i}-a_1)^q \, (v_{2_i}-a_2)^r \, (v_{3_i}-a_3)^s \qquad (A.19)$$

is the sample moment about the arbitrary points a_1,a_2,a_3. Here N is the number of observations for the sample and v_{1_i}, v_{2_i}, and v_{3_i} are the numerical values that the variates assumed for the

i th observation. In this work the sample moment about the origin and the sample moment about the mean shall be used. Then

$$m'_{qrs} = \frac{1}{N} \sum_{i=1}^{N} v_{1_i}^{\,q} \; v_{2_i}^{\,r} \; v_{3_i}^{\,s} \qquad \text{(A.20)}$$

is the sample moment about the origin. Here the mean value of v_1 is m'_{100}; the mean value of v_2 is m'_{010}; and the mean value of v_3 is m'_{001}. Therefore

$$m_{qrs} = \frac{1}{N} \sum_{i=1}^{N} (v_{1_i} - m'_{100})^q \, (v_{2_i} - m'_{010})^r$$

$$(v_{3_i} - m'_{001})^s \qquad \text{(A.21)}$$

is the sample moment about the means.

The moments uniquely determine the distribution provided that a finite upper limit exists for $(\mu'^{\,1/2n}_{2n})/2n$. According to Kendall [1958], this condition is satisfied by all the distributions arising in statistical practice. Thus a knowledge of the moments is equivalent to a knowledge of the distribution function.

sampling distribution: The distribution of the variable in the parent population has a mean, variance, and moments of higher order which are called the parameters of the population. Each sample from the population provides a distribution of the variable; and the mean, variance, and moments of higher order of the sample may be calculated. These sample moments can be looked upon as estimates of the parameters of the population. As defined before, any estimate from the sample is a statistic. Now the statistic computed from one sample will not necessarily have the same value as the identical statistic computed from another sample. Thus if a number of samples are taken, a distribution of values for a statistic is obtained. The distribution of values of the statistic obtained from a number of samples is called the sampling distribution of that statistic. The moments of the sampling distribution are called sampling moments.

characteristic function: For the univariate case, the characteristic function is defined by

$$C(t) = \int_{-\infty}^{\infty} e^{itx}\, dF \tag{A.22}$$

where t is a real variable, x is the variate, i is the pure imaginary number, and dF is the univariate distribution function. For the trivariate case, the characteristic function is defined to be

$$C(t_1,t_2,t_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ix_1t_1 + ix_2t_2 + ix_3t_3}\, dF \tag{A.23}$$

If $C(t_1,t_2,t_3)$ is expanded in powers of t_1, t_2, and t_3, then μ'_{qrs} is the coefficient of

$\frac{(it_1)^q}{q!}\ \frac{(it_2)^r}{r!}\ \frac{(it_3)^s}{s!}$ in the expansion. Thus the characteristic function is a moment generating function and

$$C(t_1,t_2,t_3) = \sum_{q,r,s=0}^{\infty} \mu'_{qrs} \frac{(it_1)^q}{q!}\ \frac{(it_2)^r}{r!}\ \frac{(it_3)^s}{s!} \tag{A.24}$$

cumulants: The cumulants are defined in terms of the population moments by the equation

$$\exp\left\{ \sum_{q,r,s=0}^{\infty} \kappa_{qrs} \frac{(it_1)^q}{q!}\ \frac{(it_2)^r}{r!}\ \frac{(it_3)^s}{s!} \right\}$$

$$= \sum_{j,k,\ell=0}^{\infty} \mu'_{jk\ell} \frac{(it_1)^j}{j!}\ \frac{(it_2)^k}{k!}\ \frac{(it_3)^\ell}{\ell!} \tag{A.25}$$

The cumulant κ_{qrs} is the coefficient of

$\frac{(it_1)^q}{q!} \frac{(it_2)^r}{r!} \frac{(it_3)^s}{s!}$ in the expansion of log

$[C(t_1,t_2,t_3)]$. By definition, κ_{000} is zero. With the exception of κ_{100}, κ_{001}, κ_{010}, the cumulants are invariant under a change of origin and in this respect they differ sharply from the moments about an arbitrary point.

symmetric function: If the observations of a sample are x_1, x_2, . . . x_n, and if the statistic computed depends explicitly on every x and its value is unchanged if we interchange any two x's, then the statistic is called a symmetric function of the observation.

k-statistics: The k-statistics are symmetric functions of the observations and are such that the first sampling moment (the mean) of k_{qrs} is the cumulant κ_{qrs}

$$E(k_{qrs}) = \kappa_{qrs} \tag{A.26}$$

Here it should be noted that whereas the sample moment m_{qrs} is the same function of the observation in a sample as μ_{qrs} is of parent population, the same relation does not hold between k_{qrs} and κ_{qrs}. Conversely, $E(m_{qrs})$ is not equal to μ_{qrs}.

variance: The variance of a distribution is the second moment about the mean. The positive square root of the variance is called the standard deviation and denoted by σ. For a statistic t, it will be convenient to write the variance of its sampling distribution as var (t). The covariance of a joint distribution is the first product moment about the mean, that is μ_{110} or μ_{101} or μ_{011}. The covariance of a joint distribution of two statistics t and u is written cov(t,u). The correlation coefficients in a sample are defined by

$$r_{12} = \frac{m_{110}}{(m_{200}m_{020})^{1/2}},$$

$$r_{13} = \frac{m_{101}}{(m_{200}m_{002})^{1/2}}, \qquad (A.27)$$

$$r_{23} = \frac{m_{011}}{(m_{020}m_{002})^{1/2}}$$

and in the population by

$$\rho_{12} = \frac{\mu_{110}}{(\mu_{200}\mu_{020})^{1/2}},$$

$$\rho_{13} = \frac{\mu_{101}}{(\mu_{200}\mu_{002})^{1/2}}, \tag{A.28}$$

$$\rho_{23} = \frac{\mu_{011}}{(\mu_{020}\mu_{002})^{1/2}}$$

So by definition, one obtains

$$\mu_{200} = \sigma_1^{\,2},$$

$$\mu_{020} = \sigma_2^{\,2}, \tag{A.29A}$$

$$\mu_{002} = \sigma_3^{\,2}$$

and

$$\mu_{110} = \rho_{12}\sigma_1\sigma_2 ,$$

$$\mu_{101} = \rho_{13}\sigma_1\sigma_3 , \tag{A-29B}$$

$$\mu_{011} = \rho_{23}\sigma_2\sigma_3$$

skewness: Asymmetrical distributions are sometimes called skewed distributions. A measure of the departure from symmetry or skewness of the distribution is given by

$$\beta_1 = \frac{\mu_3^{\,2}}{\mu_2^{\,3}} \qquad \text{(A.30)}$$

If the distribution is symmetrical it vanishes, since μ_3 vansihes. The size of μ_3 relative to $\mu_2^{3/2}$ indicates the extent of the skewness.

kurtosis: The word kurtosis means "peakedness". It arose because it was thought that a large value of the term,

$$\beta_2 = \frac{\mu_4}{\mu_2^{\,2}} \; , \qquad \text{(A.31)}$$

was associated with a high central peak in the distribution. It is now known that, in addition to the high central peak, the broadness of the tails of the distribution influences the large value of β_2. Thus a high value of the kurtosis, β_2, indicates a high central peak and broad tails for the distribution.

In general, the quantities

$$\beta_{2n+1} = \frac{\mu_3 \, \mu_{2n+3}}{\mu_2^{\,n+3}} \tag{A.32}$$

$$\beta_{2n} = \frac{\mu_{2n+2}}{\mu_2^{\,n+1}} \tag{A.33}$$

are defined to indicate the skewness or kurtosis of a distribution.

More convenient quantities than β_1 and β_2 for some purposes are

$$\gamma_1 = \frac{\mu_3'}{\mu_2^{3/2}} = \frac{\kappa_3}{\kappa_2^{3/2}} \tag{A.34}$$

and

$$\gamma_2 = \frac{\mu_4}{\mu_2^{2}} - 3 = \frac{\kappa_4}{\kappa_2^{2}} \tag{A.35}$$

Curves for which $\gamma_2 = 0$ are called mesokurtic. Those for which $\gamma_2 > 0$ are called leptokurtic; those for which $\gamma_2 < 0$ are called platykurtic.

estimator: If a statistic T, computed from a sample of observations $x_1, x_2, \ldots x_N$, is used to estimate a parameter θ which occurs in the distribution function of the parent population; then T is an estimator. The estimator T is said to be consistent if, as N increases without limit, T tends to the value θ. The estimator T is said to be unbiased if (even for finite N)

$$E(T) = \theta. \qquad (A.36)$$

The efficiency of the estimator T is given by comparing the variance of T with that of the estimator T_0 which, of all possible consistent estimators of θ, is the one with the miminum variance. Thus,

$$\text{the efficiency of } T = \text{var}\ (T_0)/\text{var}\ (T). \qquad (A.37)$$

An estimator with an efficiency of 1 is said to be the most efficient estimator of θ.

likelihood: If the continuous variate x has a p.d.f. $f(x,\theta,\theta_a)$ which depends on a parameter θ and possibly on other parameters represented jointly by θ_a, the likelihood of a set of sample values x_1, x_2, x_3 . . . x_N is defined by

$$L = f(x_1,\theta,\theta_a) \cdot f(x_2,\theta,\theta_a) \ldots \cdot f(x_N,\theta,\theta_a). \qquad (A.38)$$

The likelihood is therefore a joint probability density for the whole sample.

maximum likelihood estimator: The principle of maximum likelihood states that one should choose as an estimator of the parameter θ that statistic T which maximizes the likelihood L for variations in θ, regardless of the values of the other parameters θ_a. In practice, the logarithm of L is usually more convenient to use than L, and a value of θ which maximizes L also maximizes log L. The maximum likelihood estimator is therefore given by solving for θ the relations

$$\frac{\partial}{\partial\theta}(\log L) = 0, \quad \frac{\partial^2}{\partial\theta^2}(\log L) < 0. \tag{A.39}$$

It will be convenient to abbreviate maximum likelihood to m.ℓ..

sufficient estimators: A statistic T is a sufficient estimator of θ if T uses all the relevant information in the sample. If the likelihood function is expressed in the form

$$L = g(t,\theta)\cdot h(x_1, x_2, \ldots x_N, t, \theta) \tag{A.40}$$

or

$$\log L = \log g + \log h \tag{A.41}$$

where g is the probability density function for T and h is the p.d.f. for $x_1, x_2, \ldots x_N$, given that

$T = t$. If h does not depend on θ, then T is a sufficient estimator of θ.

properties of maximum likelihood estimators: The m.ℓ. estimator tends to normality as N increases. The m.ℓ. estimator is most-efficient. If a sufficient estimator exists for θ, the m.ℓ. estimator is sufficient. The m.ℓ. estimator is invariant under functional transformations. This means that if T is the m.ℓ. estimator of θ, and if $u(\theta)$ is a function of θ, then $u(T)$ is the m.ℓ. estimator for $u(\theta)$.

Formulas and Theorems

Relations Between Moments about an Arbitrary Point and Cumulants

If one expands the left side of equation A.25 in powers of (it_1), (it_2), and (it_3) and then equates the coefficients of $(it_1)^a\,(it_2)^b\,(it_3)^c$, the relations between moments about an arbitrary point and cumulants are obtained. The relations for the moments whose subscripts sum to four or less are given in the following set of equations. All of these relations will be of value in future research but were not necessarily used in the present work.

$$\mu'_{000} = 1$$

$$\mu'_{100} = \kappa_{100} \qquad \text{(A.42)}$$

$$\mu'_{010} = \kappa_{010}$$

$$\mu'_{001} = \kappa_{001}$$

$$\mu'_{200} = \kappa_{200} + \kappa^2_{100}$$

$$\mu'_{020} = \kappa_{020} + \kappa^2_{010}$$

$$\mu'_{002} = \kappa_{002} + \kappa^2_{001}$$

$$\mu'_{300} = \kappa_{300} + 3\,\kappa_{200}\kappa_{100} + \kappa^3_{100}$$

$$\mu'_{030} = \kappa_{030} + 3\,\kappa_{020}\kappa_{010} + \kappa^3_{010}$$

$$\mu'_{003} = \kappa_{003} + 3\,\kappa_{002}\kappa_{001} + \kappa^3_{001} \qquad \text{(A.42 Cont.)}$$

$$\mu'_{400} = \kappa_{400} + 4\,\kappa_{300}\kappa_{100} + 3\,\kappa^2_{200} + 6\,\kappa_{200}\kappa^2_{100} + \kappa^4_{100}$$

$$\mu'_{040} = \kappa_{040} + 4\,\kappa_{030}\kappa_{010} + 3\,\kappa^2_{020} + 6\,\kappa_{020}\kappa^2_{010} + \kappa^4_{010}$$

$$\mu'_{004} = \kappa_{004} + 4\,\kappa_{003}\kappa_{001} + 3\,\kappa^2_{002} + 6\,\kappa_{002}\kappa^2_{001} + \kappa^4_{001}$$

$$\mu'_{110} = \kappa_{110} + \kappa_{100}\kappa_{010}$$

$$\mu'_{101} = \kappa_{101} + \kappa_{100}\kappa_{001}$$

$$\mu'_{011} = \kappa_{011} + \kappa_{010}\kappa_{001}$$

$$\mu'_{120} = \kappa_{120} + \kappa_{100}\kappa_{020} + \kappa_{100}\kappa_{010}^2 + 2\ \kappa_{010}\kappa_{110}$$

$$\mu'_{102} = \kappa_{102} + \kappa_{100}\kappa_{002} + \kappa_{100}\kappa_{001}^2 + 2\ \kappa_{001}\kappa_{101}$$

$$\mu'_{012} = \kappa_{012} + \kappa_{010}\kappa_{002} + \kappa_{010}\kappa_{001}^2 + 2\ \kappa_{001}\kappa_{011}$$

$$\mu'_{210} = \kappa_{210} + \kappa_{200}\kappa_{010} + \kappa_{010}\kappa_{100}^2 + 2\ \kappa_{100}\kappa_{110}$$

$$\mu'_{201} = \kappa_{201} + \kappa_{200}\kappa_{001} + \kappa_{001}\kappa_{100}^2 + 2\ \kappa_{100}\kappa_{101}$$

$$\mu'_{021} = \kappa_{021} + \kappa_{020}\kappa_{001} + \kappa_{001}\kappa_{010}^2 + 2\ \kappa_{010}\kappa_{011}$$

$$\mu'_{130} = \kappa_{130} + 3\ \kappa_{020}\kappa_{110} + \kappa_{100}\kappa_{030} + 3\ \kappa_{010}\kappa_{120}$$ (A.42 Cont.)

$$+ 3\ \kappa_{100}\kappa_{010}\kappa_{020} + \kappa_{100}\kappa_{010}^3 + 3\ \kappa_{010}^2\kappa_{110}$$

$$\mu'_{103} = \kappa_{103} + 3\ \kappa_{002}\kappa_{101} + \kappa_{100}\kappa_{003} + 3\ \kappa_{001}\kappa_{102}$$

$$+ 3\ \kappa_{001}^2\kappa_{101} + 3\ \kappa_{100}\kappa_{001}\kappa_{002} + \kappa_{100}\kappa_{001}^3$$

$$\mu'_{013} = \kappa_{013} + 3\ \kappa_{002}\kappa_{011} + \kappa_{010}\kappa_{003} + 3\ \kappa_{001}\kappa_{012}$$

$$+ 3\ \kappa_{001}^2\kappa_{011} + 3\ \kappa_{010}\kappa_{001}\kappa_{002} + \kappa_{010}\kappa_{001}^3$$

$$\mu'_{310} = \kappa_{310} + 3\ \kappa_{200}\kappa_{110} + \kappa_{010}\kappa_{300} + 3\ \kappa_{100}\kappa_{210}$$

$$+ 3\kappa_{100}^2\kappa_{110} + 3\kappa_{100}\kappa_{200}\kappa_{010} + \kappa_{010}\kappa_{100}^3$$

$$\mu'_{301} = \kappa_{301} + 3\kappa_{200}\kappa_{101} + \kappa_{001}\kappa_{300} + 3\kappa_{100}\kappa_{201}$$

$$+ 3\kappa_{100}^2\kappa_{101} + 3\kappa_{100}\kappa_{200}\kappa_{001} + \kappa_{001}\kappa_{100}^3$$

$$\mu'_{031} = \kappa_{031} + 3\kappa_{020}\kappa_{011} + \kappa_{001}\kappa_{030} + 3\kappa_{010}\kappa_{021}$$

$$+ 3\kappa_{010}^2\kappa_{011} + 3\kappa_{010}\kappa_{020}\kappa_{001} + \kappa_{001}\kappa_{010}^3$$

$$\mu'_{220} = \kappa_{220} + \kappa_{200}\kappa_{020} + 2\kappa_{100}\kappa_{120} + 2\kappa_{010}\kappa_{210}$$

$$+ 4\kappa_{100}\kappa_{010}\kappa_{110} + 2\kappa_{110}^2 + \kappa_{200}\kappa_{010}^2 + \kappa_{100}^2\kappa_{020} \qquad \text{(A.42 Cont.)}$$

$$+ \kappa_{100}^2\kappa_{010}^2$$

$$\mu'_{202} = \kappa_{202} + \kappa_{200}\kappa_{002} + 2\kappa_{100}\kappa_{102} + 2\kappa_{001}\kappa_{201}$$

$$+ 4\kappa_{100}\kappa_{001}\kappa_{101} + 2\kappa_{101}^2 + \kappa_{200}\kappa_{001}^2 + \kappa_{100}^2\kappa_{002}$$

$$+ \kappa_{100}^2\kappa_{001}^2$$

$$\mu'_{022} = \kappa_{022} + \kappa_{020}\kappa_{002} + 2\kappa_{010}\kappa_{012} + 2\kappa_{001}\kappa_{021}$$

$$+ 4\kappa_{010}\kappa_{001}\kappa_{011} + 2\kappa_{011}^2 + \kappa_{020}\kappa_{001}^2 + \kappa_{010}^2\kappa_{002} + \kappa_{010}^2\kappa_{001}^2$$

$$\mu'_{111} = \kappa_{111} + \kappa_{100}\kappa_{011} + \kappa_{010}\kappa_{101} + \kappa_{001}\kappa_{110} + \kappa_{100}\kappa_{010}\kappa_{001}$$

$$\mu'_{112} = \kappa_{112} + \kappa_{010}\kappa_{102} + \kappa_{100}\kappa_{012} + \kappa_{002}\kappa_{110} + 2\ \kappa_{001}\kappa_{111}$$

$$+ 2\ \kappa_{101}\kappa_{011} + \kappa_{100}\kappa_{010}\kappa_{002} + 2\ \kappa_{100}\kappa_{001}\kappa_{011}$$

$$+ 2\ \kappa_{010}\kappa_{001}\kappa_{101} + \kappa_{100}\kappa_{010}\kappa^2_{001} + \kappa^2_{001}\kappa_{110}$$

$$\mu'_{121} = \kappa_{121} + \kappa_{100}\kappa_{021} + \kappa_{001}\kappa_{120} + \kappa_{020}\kappa_{101} + 2\ \kappa_{010}\kappa_{111}$$

$$+ 2\ \kappa_{110}\kappa_{011} + \kappa_{100}\kappa_{020}\kappa_{001} + 2\ \kappa_{100}\kappa_{010}\kappa_{011} \qquad \text{(A.42 Cont.)}$$

$$+ 2\ \kappa_{010}\kappa_{001}\kappa_{110} + \kappa_{100}\kappa^2_{010}\kappa_{001} + \kappa^2_{010}\kappa_{101}$$

$$\mu'_{211} = \kappa_{211} + \kappa_{200}\kappa_{011} + \kappa_{001}\kappa_{210} + \kappa_{010}\kappa_{201} + 2\ \kappa_{100}\kappa_{111}$$

$$+ 2\ \kappa_{110}\kappa_{101} + \kappa_{200}\kappa_{010}\kappa_{001} + 2\ \kappa_{100}\kappa_{010}\kappa_{101}$$

$$+ 2\ \kappa_{100}\kappa_{001}\kappa_{110} + \kappa^2_{100}\kappa_{010}\kappa_{001} + \kappa^2_{100}\kappa_{011}$$

Relations Between Moments about the Means and Cumulants

In the case of moments about the mean, the first order cumulants become zero. Thus

$$\kappa_{100} = 0, \qquad \kappa_{010} = 0, \qquad \kappa_{001} = 0 \qquad \text{(A.43)}$$

One may obtain the relations between the moments about the mean and the cumulants from equation A.42 by substituting the values for κ_{100}, κ_{010}, κ_{001} and by simply removing the primes from the moments. When this is done one has the following relations:

$$\mu_{000} = 1$$

$$\mu_{100} = 0 \quad \mu_{010} = 0 \quad \mu_{001} = 0$$

$$\mu_{200} = \kappa_{200} \quad \mu_{020} = \kappa_{020} \quad \mu_{002} = \kappa_{002}$$

$$\mu_{300} = \kappa_{300} \quad \mu_{030} = \kappa_{030} \quad \mu_{003} = \kappa_{003}$$

$$\mu_{400} = \kappa_{400} + 3\ \kappa_{200}^2$$

$$\mu_{040} = \kappa_{040} + 3\ \kappa_{020}^2 \tag{A.44}$$

$$\mu_{004} = \kappa_{004} + 3\ \kappa_{002}^2$$

$$\mu_{110} = \kappa_{110} \quad \mu_{101} = \kappa_{101} \quad \mu_{011} = \kappa_{011}$$

$$\mu_{120} = \kappa_{120} \quad \mu_{102} = \kappa_{102} \quad \mu_{012} = \kappa_{012}$$

$$\mu_{210} = \kappa_{210} \quad \mu_{201} = \kappa_{201} \quad \mu_{021} = \kappa_{021}$$

$$\mu_{130} = \kappa_{130} + 3\ \kappa_{020}\kappa_{110}$$

$$\mu_{103} = \kappa_{103} + 3\,\kappa_{002}\kappa_{101}$$

$$\mu_{013} = \kappa_{013} + 3\,\kappa_{002}\kappa_{011}$$

$$\mu_{310} = \kappa_{310} + 3\,\kappa_{200}\kappa_{110}$$

$$\mu_{301} = \kappa_{301} + 3\,\kappa_{200}\kappa_{101}$$

$$\mu_{031} = \kappa_{031} + 3\,\kappa_{020}\kappa_{011}$$

$$\mu_{220} = \kappa_{220} + \kappa_{200}\kappa_{020} + 2\,\kappa_{110}^2$$

$$\mu_{202} = \kappa_{202} + \kappa_{200}\kappa_{002} + 2\,\kappa_{101}^2 \qquad \text{(A.44 Cont.)}$$

$$\mu_{022} = \kappa_{022} + \kappa_{020}\kappa_{002} + 2\,\kappa_{011}^2$$

$$\mu_{111} = \kappa_{111}$$

$$\mu_{112} = \kappa_{112} + \kappa_{002}\kappa_{110} + 2\,\kappa_{101}\kappa_{011}$$

$$\mu_{121} = \kappa_{121} + \kappa_{020}\kappa_{101} + 2\,\kappa_{110}\kappa_{011}$$

$$\mu_{211} = \kappa_{211} + \kappa_{200}\kappa_{011} + 2\,\kappa_{110}\kappa_{101}$$

Trivariate k-statistics in Terms of Power Sums

Consider the case of the trivariate distribution where the variates are v_1, v_2, and v_3. If $W_{i,j,K,\ell}$ refers to the products $v_i v_j v_K v_\ell$ summed over the sample of size N (here i,j,K,ℓ = 1 or 2 or 3), one has for the k-statistics

$$k_i = W_i/N \tag{A.45}$$

$$k_{ij} = (N\, W_{ij} - W_i W_j)/\, N^{[2]} \tag{A.46}$$

$$k_{ijK} = (N^2\, W_{ijK} - N \overset{3}{\Sigma}\, W_i W_{jK} + 2\, W_i W_j W_K)/N^{[3]} \tag{A.47}$$

$$k_{ijK\ell} = \{N^2(N+1)\, W_{ijK\ell} - N(N+1) \overset{4}{\Sigma}\, W_i W_{jK\ell}$$

$$- N(N-1) \overset{3}{\Sigma}\, W_{ij} W_{K\ell} + 2N \overset{6}{\Sigma}\, W_i W_j W_{K\ell} - 6\, W_i W_j W_K W_\ell\}/N^{[4]} \tag{A.48}$$

where $$N^{[P]} = N(N-1)\ \ldots\ (N-P+1), \tag{A.49}$$

where here the indices of the k's refer to the indices of the products of the variates, and where the summations occur over all distinguishable ways of grouping the indices; the number of ways is shown above the summation sign. Thus the summation $\overset{3}{\Sigma}$ in the equation for k_{ijK} gives the terms: $W_i W_{jK}$, $W_j W_{iK}$, $W_K W_{ij}$. Similarly in the equation for $k_{ijK\ell}$, one has

$$\sum^{4} \rightarrow \begin{array}{l} W_i\, W_{jK\ell} \\ W_j\, W_{iK\ell} \\ W_K\, W_{ij\ell} \\ W_\ell\, W_{ijK} \end{array} ,$$

and

$$\sum^{3} \rightarrow \begin{array}{l} W_{ij}W_{K\ell} \\ W_{iK}W_{j\ell} \\ W_{i\ell}W_{jK} \end{array} ,$$

and

$$\sum^{6} \rightarrow \begin{array}{l} W_iW_jW_{K\ell} \\ W_iW_KW_{j\ell} \\ W_iW_\ell W_{jK} \\ W_jW_KW_{i\ell} \\ W_jW_\ell W_{ik} \\ W_KW_\ell W_{ij} \end{array}$$

Note the difference in meaning of the symbol $\sum^{3}$ when used in equations A.47 and A.48. If the example in which the ith and jth variates are both equal to v_1, one obtains

$$k_{200} = (N\, S_{200} - S_{100}^2) \;/\; \{N(N-1)\}$$

where S_{ijK} is called a power sum and refers to the summation over the sample of the i th power of v_1 times the j th power of v_2 times the K th power of v_3, and where here the subscripts of the k's are the powers of the variates and these are the natural subscripts of the k's. Thus by amalgamation of the psuedo indices in equations A.45, A.46, A.47, and A.48, one obtains formulas for the trivariate k-statistics in terms of power sums. All of these relations will be of value in future research but were not necessarily used in the present work. The first few of these up to order four are given in the following relations:

$$k_{100} = \frac{S_{100}}{n} \qquad k_{010} = \frac{S_{010}}{n} \qquad k_{001} = \frac{S_{001}}{n}$$

$$k_{110} = [nS_{110} - S_{100}S_{010}]/[n(n-1)] \qquad k_{011} = [nS_{011} - S_{010}S_{001}]/[n(n-1)]$$

$$k_{101} = [nS_{101} - S_{100}S_{001}]/[n(n-1)]$$

$$k_{200} = [nS_{200} - S_{100}^2]/[n(n-1)]$$

$$k_{020} = [nS_{020} - S_{010}^2]/[n(n-1)] \tag{A.50}$$

$$k_{002} = [nS_{002} - S_{001}^2]/[n(n-1)]$$

$$k_{111} = [n^2S_{111} - n(S_{100}S_{011} + S_{010}S_{101} + S_{001}S_{110})$$

$$+ 2\, S_{100}S_{010}S_{001}]/[n(n-1)(n-2)]$$

$$k_{210} = [n^2 s_{210} - n(2\ s_{100}s_{110} + s_{010}s_{200}) + 2\ s_{100}^2 s_{010}]/n^{[3]}$$

$$k_{120} = [n^2 s_{120} - n(s_{100}s_{020} + 2\ s_{010}s_{110}) + 2\ s_{100}s_{010}^2]/n^{[3]}$$

$$k_{201} = [n^2 s_{201} - n(2\ s_{100}s_{101} + s_{001}s_{200}) + 2\ s_{100}^2 s_{001}]/n^{[3]}$$

$$k_{102} = [n^2 s_{102} - n(s_{100}s_{002} + 2\ s_{001}s_{101}) + 2\ s_{100}s_{001}^2]/n^{[3]}$$

$$k_{021} = [n^2 s_{021} - n(2\ s_{010}s_{011} + s_{001}s_{020}) + 2\ s_{010}^2 s_{001}]/n^{[3]}$$

$$k_{012} = [n^2 s_{012} - n(s_{010}s_{002} + 2\ s_{001}s_{011}) + 2\ s_{010}s_{001}^2]/n^{[3]}$$

$$k_{300} = [n^2 s_{300} - n(3\ s_{100}s_{200}) + 2\ s_{100}^3]/n^{[3]}$$

$$k_{030} = [n^2 s_{030} - n(3\ s_{010}s_{020}) + 2\ s_{010}^3]/n^{[3]} \qquad \text{(A.50 Cont.)}$$

$$k_{003} = [n^2 s_{003} - n(3\ s_{001}s_{002}) + 2\ s_{001}^3]/n^{[3]}$$

$$k_{400} = \{n^2(n+1)\ s_{400} - n(n+1)[4\ s_{100}s_{300}] - n(n-1)[3\ s_{200}^2]$$

$$+ 2\ n[6\ s_{100}^2 s_{200}] - 6 s_{100}^4\}/[n(n-1)(n-2)(n-3)]$$

$$k_{040} = \{n^2(n+1)\ s_{040} - n(n+1)[4\ s_{010}s_{030}] - n(n-1)\ [3\ s_{020}^2]$$

$$+ 2\ n[6\ s_{010}^2 s_{020}] - 6\ s_{010}^4\}/n^{[4]}$$

$$k_{004} = \{n^2(n+1)\ s_{004} - n(n+1)\ [4\ s_{001}s_{003}] - n(n-1)\ [3\ s_{002}^2]$$

$$+ 2\ n[6\ s_{001}^2 s_{002}] - 6\ s_{001}^4\}/n^{[4]}$$

$$k_{310} = \{n^2(n+1)\ s_{310} - n(n+1)\ [3\ s_{100}s_{210} + s_{010}s_{300}]$$

$$- n(n-1)\ [3\ s_{110}s_{200}] + 2\ n[3\ s_{100}^2 s_{110} + 3\ s_{100}s_{010}s_{200}]$$

$$- 6\ s_{100}^3 s_{010}\}n^{[4]}$$

$$k_{130} = \{n^2(n+1)\ s_{130} - n(n+1)\ [s_{100}s_{030} + 3\ s_{010}s_{120}]$$

$$- n(n-1)\ [3\ s_{110}s_{020}] + 2n[3\ s_{100}s_{010}s_{020} + 3\ s_{010}^2 s_{110}]$$

$$- 6\ s_{100}s_{010}^3\}/n^{[4]}$$

(A.50 Cont.)

$$k_{301} = \{n^2(n+1)\ s_{301} - n(n+1)\ [3\ s_{100}s_{201} + s_{001}s_{300}]$$

$$- n(n-1)\ [3\ s_{200}s_{101}] + 2n[3\ s_{100}^2 s_{101} + 3\ s_{100}s_{001}s_{200}]$$

$$- 6\ s_{100}^3 s_{001}\}/n^{[4]}$$

$$k_{103} = \{n^2(n+1)\ s_{103} - n(n+1)\ [s_{100}s_{003} + 3\ s_{001}s_{102}]$$

$$- n(n-1)[3\ s_{101}s_{002}] + 2n[3\ s_{100}s_{001}s_{002} + 3\ s_{001}^2 s_{101}]$$

$$- 6\ s_{100}s_{001}^3\}/n^{[4]}$$

$$k_{031} = \{n^2(n+1)\ s_{031} - n(n+1)\ [3\ s_{010}s_{021} + s_{001}s_{030}]$$

$$-n(n-1)\ [3\ s_{020}s_{011}] + 2n[3\ s_{010}^2 s_{011} + 3\ s_{010}s_{001}s_{020}]$$

$$- 6\ s_{010}^3 s_{001}\}/n^{[4]}$$

$$k_{013} = \{n^2(n+1)\ s_{013} - n(n+1)\ [s_{010}s_{003} + 3\ s_{001}s_{012}]$$

$$- n(n-1)\ [3\ s_{011}s_{002}] + 2n[3\ s_{010}s_{001}s_{002} + 3\ s_{001}^2 s_{011}]$$

$$- 6\ s_{010}s_{001}^3\}/n^{[4]}$$

(A.50 Cont.)

$$k_{220} = \{n^2(n+1)\ s_{220} - n(n+1)\ [2\ s_{100}s_{120} + 2\ s_{010}s_{210}]$$

$$- n(n-1)\ [s_{200}s_{020} + 2\ s_{110}^2] + 2\ n\ [s_{100}^2 s_{020}$$

$$+ 4\ s_{100}s_{010}s_{110} + s_{010}^2 s_{200}] - 6\ s_{100}^2 s_{010}^2\}/n^{[4]}$$

$$k_{202} = \{n^2(n+1)\ s_{202} - n(n+1)\ [2\ s_{100}s_{102} + 2\ s_{001}s_{201}]$$

$$- n(n-1)\ [s_{200}s_{002} + 2\ s_{101}^2] + 2n\ [s_{100}^2 s_{002}$$

$$+ 4\ s_{100}s_{001}s_{101} + s_{001}^2 s_{200}] - 6\ s_{100}^2 s_{001}^2\}/n^{[4]}$$

$$k_{022} = \{n^2(n+1)\ s_{022} - n(n+1)\ [2\ s_{010}s_{012} + 2\ s_{001}s_{021}]$$

$$- n(n-1)\ [s_{020}s_{002} + 2\ s_{011}^2] + 2n\ [s_{010}^2 s_{002}$$

$$+ 4\ s_{010}s_{001}s_{011} + s_{001}^2 s_{020}] - 6\ s_{010}^2 s_{001}^2\}/n^{[4]}$$

$$k_{211} = \{n^2(n+1)\ s_{211} - n(n+1)\ [2\ s_{100}s_{111} + s_{010}s_{201} + s_{001}s_{210}]$$

$$- n(n-1)\ [s_{200}s_{011} + 2\ s_{110}s_{101}] + 2n\ [s_{100}^2 s_{011}$$

$$+ 2\ s_{100}s_{010}s_{101} + 2\ s_{100}s_{001}s_{110} + s_{010}s_{001}s_{200}]$$

$$- 6\ s_{100}^2 s_{010}s_{001}\}/n^{[4]}$$

(A.50 Cont.)

$$k_{121} = \{n^2(n+1)\ s_{121} - n(n+1)\ [s_{100}s_{021} + 2\ s_{010}s_{111}$$

$$+ s_{001}s_{120}] - n(n-1)\ [2\ s_{110}s_{011} + s_{101}s_{020}]$$

$$+ 2n\ [s_{010}^2 s_{101} + 2\ s_{100}s_{010}s_{011} + 2\ s_{010}s_{001}s_{110}$$

$$+ s_{100}s_{001}s_{020}] - 6\ s_{100}s_{010}^2 s_{001}\}/n^{[4]}$$

$$k_{112} = \{n^2(n+1)\ s_{112} - n(n+1)\ [s_{100}s_{012} + s_{010}s_{102}$$

$$+ 2\ s_{001}s_{111}] - n(n-1)\ [s_{110}s_{002} + 2\ s_{101}s_{011}]$$

$$+ 2n[S_{100}S_{010}S_{002} + 2\, S_{100}S_{001}S_{011} + 2\, S_{010}S_{001}S_{101} \qquad \text{(A.50 Cont.)}$$

$$+ S_{001}^{2}S_{110}] -6\, S_{100}S_{010}S_{001}^{2}\}/n^{[4]}$$

In the above equation A.50, n is equal to N.

Relations Between the k-statistics and the Sample Moments About Zero.

By considering the written definition of S_{qrs} and the definition of m'_{qrs} in equation A.20, it is obvious that

$$m'_{qrs} = \frac{1}{n}\, S_{qrs} \qquad \text{(A.51)}$$

where n is the number of observations in the sample.

Using this equation, one can determine the relations between the k-statistics and the sample moments about zero. One then has the following relations (All of these relations will be of value in future research but were not necessarily used in the present work).

$$k_{100} = m'_{100} \qquad k_{010} = m'_{010} \qquad k_{001} = m'_{001}$$

$$k_{110} = [\frac{n}{n-1}]\, [m'_{110} - m'_{100}m'_{010}]$$

$$k_{101} = [\frac{n}{n-1}]\, [m'_{101} - m'_{100}m'_{001}] \qquad \text{(A.52)}$$

$$k_{011} = [\frac{n}{n-1}]\, [m'_{011} - m'_{010}m'_{001}]$$

$$k_{200} = [\frac{n}{n-1}]\, [m'_{200} - {m'_{100}}^{2}]$$

$$k_{020} = [\frac{n}{n-1}] [m'_{020} - m'^{2}_{010}]$$

$$k_{002} = [\frac{n}{n-1}] [m'_{002} - m'^{2}_{001}]$$

$$k_{111} = [\frac{n^2}{(n-1)(n-2)}] [m'_{111} - (m'_{100}m'_{011} + m'_{010}m'_{101} + m'_{001}m'_{110})$$

$$+ 2\, m'_{100}m'_{010}m'_{001}]$$

$$k_{300} = [\frac{n^2}{(n-1)(n-2)}] [m'_{300} - 3\, m'_{100}m'_{200} + 2\, m'^{3}_{100}]$$

$$k_{030} = [\frac{n^2}{(n-1)(n-2)}] [m'_{030} - 3\, m'_{010}m'_{020} + 2\, m'^{3}_{010}]$$

$$k_{003} = [\frac{n^2}{(n-1)(n-2)}] [m'_{003} - 3\, m'_{001}m'_{002} + 2\, m'^{3}_{001}]$$

$$k_{210} = [\frac{n^2}{(n-1)(n-2)}] [m'_{210} - 2\, m'_{100}m'_{110} - m'_{010}m'_{200}$$

$$+ 2\, m'^{2}_{100}m'_{010}] \qquad \text{(A.52 Cont)}$$

$$k_{120} = [\frac{n^2}{(n-1)(n-2)}] [m'_{120} - m'_{100}m'_{020} - 2\, m'_{010}m'_{110}$$

$$+ 2\, m'_{100}m'^{2}_{010}]$$

$$k_{201} = [\frac{n^2}{(n-1)(n-2)}] [m'_{201} - 2\, m'_{100}m'_{101} - m'_{001}m'_{200}$$

$$+ 2\, m'^{2}_{100}m'_{001}]$$

$$k_{102} = [\frac{n^2}{(n-1)(n-2)}] [m'_{102} - m'_{100}m'_{002} - 2\ m'_{001}m'_{101}$$

$$+ 2\ m'_{100}m'^{2}_{001}]$$

$$k_{021} = [\frac{n^2}{(n-1)(n-2)}] [m'_{021} - 2\ m'_{010}m'_{011} - m'_{001}m'_{020}$$

$$+ 2\ m'^{2}_{010}m'_{001}]$$

$$k_{012} = [\frac{n^2}{(n-1)(n-2)}] [m'_{012} - m'_{010}m'_{002} - 2\ m'_{001}m'_{011}$$

$$+ 2\ m'_{010}m'^{2}_{001}]$$

$$k_{400} = [\frac{n^2}{(n-1)(n-2)(n-3)}] [(n+1)\ m'_{400} - 4(n+1)\ m'_{100}m'_{300}$$

$$- 3(n-1)\ m'^{2}_{200} + 12\ n\ m'^{2}_{100}m'_{200} - 6\ n\ m'^{4}_{100}] \qquad \text{(A.52 Cont.)}$$

$$k_{040} = [\frac{n^2}{(n-1)(n-2)(n-3)}] [(n+1)\ m'_{040} - 4(n+1)\ m'_{010}m'_{030}$$

$$- 3(n-1)\ m'^{2}_{020} + 12\ n\ m'^{2}_{010}m'_{020} - 6\ n\ m'^{4}_{010}]$$

$$k_{004} = [\frac{n^2}{(n-1)(n-2)(n-3)}] [(n+1)\ m'_{004} - 4(n+1)\ m'_{001}m'_{003}$$

$$- 3(n-1)\ m'^{2}_{002} + 12\ n\ m'^{2}_{001}m'_{002} - 6\ n\ m'^{4}_{001}]$$

$$k_{310} = [\frac{n^2}{(n-1)(n-2)(n-3)}] [(n+1)\ m'_{310} - 3(n+1)\ m'_{100}m'_{210}$$

$$-(n+1)\ m'_{010}m'_{300} - 3(n-1)\ m'_{110}m'_{200} + 6n\ m'^{2}_{100}m'_{110}$$

$$+ 6\ n\ m'_{100}m'_{010}m'_{200} - 6\ n\ m'^{3}_{100}\ m'_{010}]$$

$$k_{130} = [\frac{n^2}{(n-1)(n-2)(n-3)}]\ [(n+1)\ m'_{130} - (n+1)\ m'_{100}m'_{030}$$

$$-3(n+1)\ m'_{010}m'_{120} - 3(n-1)\ m'_{110}m'_{020} + 6\ n\ m'_{100}m'_{010}m'_{020}$$

$$+ 6\ n\ m'^{2}_{010}m'_{110} - 6\ n\ m'_{100}m'^{3}_{010}]$$

$$k_{301} = [\frac{n^2}{(n-1)(n-2)(n-3)}]\ [(n+1)\ m'_{301} - 3(n+1)\ m'_{100}m'_{201}$$

$$-(n+1)\ m'_{001}m'_{300} - 3(n-1)\ m'_{200}m'_{101} + 6\ n\ m'^{2}_{100}m'_{101}$$

(A.52 Cont.)

$$+ 6\ n\ m'_{100}m'_{001}m'_{200} - 6\ n\ m'^{3}_{100}m'_{001}]$$

$$k_{103} = [\frac{n^2}{(n-1)(n-2)(n-3)}]\ [(n+1)\ m'_{103} - (n+1)\ m'_{100}m'_{003}$$

$$-3(n+1)\ m'_{001}m'_{102} - 3(n-1)\ m'_{101}m'_{002} - 6\ n\ m'_{100}m'^{3}_{001}$$

$$+ 6\ n\ m'_{100}m'_{001}m'_{002} + 6\ n\ m'^{2}_{001}m'_{101}]$$

$$k_{031} = [\frac{n^2}{(n-1)(n-2)(n-3)}]\ [(n+1)\ m'_{031} - 3(n+1)\ m'_{010}m'_{021}$$

$$-(n+1)\ m'_{001}m'_{030} - 3(n-1)\ m'_{020}m'_{011} + 6\ n\ m'^{2}_{010}m'_{011}$$

$$+ 6\ n\ m'_{010}m'_{001}m'_{020} - 6\ n\ m'^{3}_{010}m'_{001}]$$

$$k_{013} = [\frac{n^2}{(n-1)(n-2)(n-3)}]\ [(n+1)\ m'_{013} - (n+1)\ m'_{010}m'_{003}$$

$$- 3(n+1)\ m'_{001}m'_{012} - 3(n-1)\ m'_{011}m'_{002} + 6n\ m'_{010}m'_{001}m'_{002}$$

$$+ 6\ n\ m'^{2}_{001}m'_{011} - 6\ n\ m'_{010}m'^{3}_{001}]$$

$$k_{220} = [\frac{n^2}{(n-1)(n-2)(n-3)}]\ [(n+1)\ m'_{220} - 2(n+1)\ m'_{100}m'_{120}$$

$$- 2(n+1)\ m'_{010}m'_{210} -(n-1)\ m'_{200}m'_{020} -2(n-1)\ m'^{2}_{110}$$

$$+ 2\ n\ m'^{2}_{100}m'_{020} + 8\ n\ m'_{100}m'_{010}m'_{110} + 2n\ m'^{2}_{010}m'_{200}$$

(A.52 Cont.)

$$- 6\ n\ m'^{2}_{100}m'^{2}_{010}]$$

$$k_{202} = [\frac{n^2}{(n-1)(n-2)(n-3)}]\ [(n+1)\ m'_{202} - 2(n+1)\ m'_{100}m'_{102}$$

$$- 2(n+1)\ m'_{001}m'_{201} - (n-1)\ m'_{200}m'_{002} -2(n-1)\ m'^{2}_{101}$$

$$+ 2\ n\ m'^{2}_{100}m'_{002} + 8\ n\ m'_{100}m'_{001}m'_{101} + 2\ n\ m'^{2}_{001}m'_{200}$$

$$- 6\ n\ m'^{2}_{100}m'^{2}_{001}]$$

$$k_{022} = [\frac{n^2}{(n-1)(n-2)(n-3)}]\ [(n+1)\ m'_{022} - 2(n+1)\ m'_{010}m'_{012}$$

$$- 2(n+1)\ m'_{001}m'_{021} - (n-1)\ m'_{020}m'_{002} - 2(n-1)\ m'^{2}_{011}$$

$$+ 2\ n\ m'^{2}_{010}m'_{002} + 8\ n\ m'_{010}m'_{001}m'_{011} + 2\ n\ m'^{2}_{001}m'_{020}$$

$$- 6\ n\ m'^{2}_{010}m'^{2}_{001}]$$

$$k_{211} = [\frac{n^2}{(n-1)(n-2)(n-3)}]\ [(n+1)\ m'_{211} - 2(n+1)\ m'_{100}m'_{111}$$

$$- (n+1)\ m'_{010}m'_{201} - (n+1)\ m'_{001}m'_{210} - (n-1)\ m'_{200}m'_{011}$$

$$-2(n-1)\ m'_{110}m'_{101} + 2\ n\ m'^{2}_{100}m'_{011} + 4\ n\ m'_{100}m'_{010}m'_{101}$$

$$+ 4\ n\ m'_{100}m'_{001}m'_{110} + 2\ n\ m'_{010}m'_{001}m'_{200}$$

$$- 6\ n\ m'^{2}_{100}m'_{010}m'_{001}] \qquad \text{(A.52 Cont.)}$$

$$k_{121} = [\frac{n^2}{(n-1)(n-2)(n-3)}]\ [(n+1)\ m'_{121} - (n+1)\ m'_{100}m'_{021}$$

$$- 2(n+1)\ m'_{010}m'_{111} - (n+1)\ m'_{001}m'_{120} - 2(n-1)\ m'_{110}m'_{011}$$

$$- (n-1)\ m'_{101}m'_{020} + 2\ n\ m'^{2}_{010}m'_{101} + 4\ n\ m'_{100}m'_{010}m'_{011}$$

$$+ 4\ n\ m'_{010}m'_{001}m'_{110} + 2\ n\ m'_{100}m'_{001}m'_{020}$$

$$- 6\ n\ m'_{100}m'^{2}_{010}m'_{001}]$$

$$k_{112} = \left[\frac{n^2}{(n-1)(n-2)(n-3)}\right] [(n+1)\, m'_{112} - (n+1)\, m'_{100}m'_{012}$$

$$-(n+1)\, m'_{010}m'_{102} - 2(n+1)\, m'_{001}m'_{111} - (n-1)\, m'_{110}m'_{002}$$

$$-2(n-1)\, m'_{101}m'_{011} + 2\, n\, m'_{100}m'_{010}m'_{002} \qquad \text{(A.52 Cont.)}$$

$$+ 4n\, m'_{100}m'_{001}m'_{011} + 4n\, m'_{010}m'_{001}m'_{101} + 2n\, m'^{2}_{001}m'_{110}$$

$$- 6n\, m'_{100}m'_{010}m'^{2}_{001}]$$

Relations between the Moments about the Mean and the Moments about Zero.

By expanding equation A.21 and recognizing the terms in the form of the right hand side of A.20, one can determine the relations between the moments about the mean and the moments about zero. For example, consider the following

$$m_{300} = \sum_{i=1}^{N} (v_{1_i} - m'_{100})^3\, (v_{2_i} - m'_{010})^0\, (v_{3_i} - m'_{011})^0$$

$$= \sum_{i=1}^{N} \{v_{1_i}^{\,3} - 3\, v_{1_i}^{\,2}\, m'_{100} + 3\, v_{1_i}\, m'^{2}_{100} - m'^{3}_{100}\}$$

$$= [\sum_{i=1}^{N} v_{1_i}^{\,3}] - 3[\sum_{i=1}^{N} v_{1_i}^{\,2}]\, m'_{100} + 3[\sum_{i=1}^{N} v_{1_i}]\, m'^{2}_{100} - m'^{3}_{100}$$

But the terms in brackets are forms of the right hand side of A.20, so

$$m_{300} = m'_{300} - 3\ m'_{200}m'_{100} + 3\ m'_{100}m'_{100}{}^{2} - m'_{100}{}^{3}$$

where on combining terms one has,

$$m_{300} = m'_{300} - 3\ m'_{200}m'_{100} + 2\ m'_{100}{}^{3}$$

In a similar manner, one arrives at the following relations:

$$m_{100} = 0 \qquad m_{010} = 0 \qquad m_{001} = 0$$

$$m_{200} = m'_{200} - m'_{100}{}^{2} \qquad m_{110} = m'_{110} - m'_{100}m'_{010}$$

$$m_{020} = m'_{020} - m'_{010}{}^{2} \qquad m_{101} = m'_{101} - m'_{100}m'_{001} \tag{A.53}$$

$$m_{002} = m'_{002} - m'_{001}{}^{2} \qquad m_{011} = m'_{011} - m'_{010}m'_{001}$$

$$m_{300} = m'_{300} - 3\ m'_{200}m'_{100} + 2\ m'_{100}{}^{3}$$

$$m_{030} = m'_{030} - 3\ m'_{020}m'_{010} + 2\ m'_{010}{}^{3}$$

$$m_{003} = m'_{003} - 3\ m'_{002}m'_{001} + 2\ m'_{001}{}^{3}$$

$$m_{111} = m'_{111} - m'_{100}m'_{011} - m'_{010}m'_{101} - m'_{001}m'_{110} + 2\ m'_{100}m'_{010}m'_{001}$$

$$m_{210} = m'_{210} - 2\, m'_{100}m'_{110} - m'_{200}m'_{010} + 2\, m'^{2}_{100}m'_{010}$$

$$m_{120} = m'_{120} - 2\, m'_{010}m'_{110} - m'_{100}m'_{020} + 2\, m'_{100}m'^{2}_{010}$$

$$m_{201} = m'_{201} - 2\, m'_{100}m'_{101} - m'_{200}m'_{001} + 2\, m'^{2}_{100}m'_{001}$$

$$m_{102} = m'_{102} - 2\, m'_{001}m'_{101} - m'_{100}m'_{002} + 2\, m'_{100}m'^{2}_{001}$$

$$m_{021} = m'_{021} - 2\, m'_{010}m'_{011} - m'_{001}m'_{020} + 2\, m'^{2}_{010}m'_{001}$$

$$m_{012} = m'_{012} - 2\, m'_{001}m'_{011} - m'_{010}m'_{002} + 2\, m'_{010}m'^{2}_{001}$$

$$m_{400} = m'_{400} - 4\, m'_{100}m'_{300} + 6\, m'^{2}_{100}m'_{200} - 3\, m'^{4}_{100} \qquad \text{(A.53 Cont.)}$$

$$m_{040} = m'_{040} - 4\, m'_{010}m'_{030} + 6\, m'^{2}_{010}m'_{020} - 3\, m'^{4}_{010}$$

$$m_{004} = m'_{004} - 4\, m'_{001}m'_{003} + 6\, m'^{2}_{001}m'_{002} - 3\, m'^{4}_{001}$$

$$m_{310} = m'_{310} - 3\, m'_{100}m'_{210} - m'_{010}m'_{300} + 3\, m'^{2}_{100}m'_{110} + 3\, m'_{100}m'_{010}m'_{200} - 3\, m'^{3}_{100}m'_{010}$$

$$m_{130} = m'_{130} - 3\, m'_{010}m'_{120} - m'_{100}m'_{030} + 3\, m'^{2}_{010}m'_{110} + 3\, m'_{100}m'_{010}m'_{020} - 3\, m'_{100}m'^{3}_{010}$$

$$m_{301} = m'_{301} - 3\, m'_{100}m'_{201} - m'_{001}m'_{300} + 3\, m'_{100}{}^{2}m'_{101}$$

$$+ 3\, m'_{100}m'_{001}m'_{200} - 3\, m'_{100}{}^{3}m'_{001}$$

$$m_{103} = m'_{103} - 3\, m'_{001}m'_{102} - m'_{100}m'_{003} + 3\, m'_{001}{}^{2}m'_{101}$$

$$+ 3\, m'_{100}m'_{001}m'_{002} - 3\, m'_{100}m'_{001}{}^{3}$$

$$m_{031} = m'_{031} - 3\, m'_{010}m'_{021} - m'_{001}m'_{030} + 3\, m'_{010}{}^{2}m'_{011}$$

$$+ 3\, m'_{010}m'_{001}m'_{020} - 3\, m'_{010}{}^{3}m'_{001}$$

$$m_{013} = m'_{013} - 3\, m'_{001}m'_{012} - m'_{010}m'_{003} + 3\, m'_{001}{}^{2}m'_{011} \qquad \text{(A.53 Cont.)}$$

$$+ 3\, m'_{010}m'_{001}m'_{002} - 3\, m'_{010}m'_{001}{}^{3}$$

$$m_{220} = m'_{220} - 2\, m'_{100}m'_{120} - 2\, m'_{010}m'_{210} + m'_{100}{}^{2}m'_{020}$$

$$+ 4\, m'_{100}m'_{010}m'_{110} + m'_{010}{}^{2}m'_{200} - 3\, m'_{100}{}^{2}m'_{010}{}^{2}$$

$$m_{202} = m'_{202} - 2\, m'_{100}m'_{102} - 2\, m'_{001}m'_{201} + m'_{100}{}^{2}m'_{002}$$

$$+ 4\, m'_{100}m'_{001}m'_{101} + m'_{001}{}^{2}m'_{200} - 3\, m'_{100}{}^{2}m'_{001}{}^{2}$$

$$m_{022} = m'_{022} - 2\, m'_{010}m'_{012} - 2\, m'_{001}m'_{021} + m'_{010}{}^{2}m'_{002}$$

$$+ 4\, m'_{010} m'_{001} m'_{011} + m'^{2}_{001} m'_{020} - 3\, m'^{2}_{010} m'^{2}_{001}$$

$$m_{211} = m'_{211} - 2\, m'_{100} m'_{111} - m'_{010} m'_{201} - m'_{001} m'_{210}$$

$$+ m'^{2}_{100} m'_{011} + 2\, m'_{100} m'_{010} m'_{101} + 2\, m'_{100} m'_{001} m'_{110}$$

$$+ m'_{010} m'_{001} m'_{200} - 3\, m'^{2}_{100} m'_{010} m'_{001}$$

$$m_{121} = m'_{121} - 2\, m'_{010} m'_{111} - m'_{100} m'_{021} - m'_{001} m'_{120}$$

$$+ m'^{2}_{010} m'_{101} + 2\, m'_{100} m'_{010} m'_{011} + 2\, m'_{010} m'_{001} m'_{110} \qquad \text{(A.53 Cont.)}$$

$$+ m'_{100} m'_{001} m'_{020} - 3\, m'_{100} m'^{2}_{010} m'_{001}$$

$$m_{112} = m'_{112} - 2\, m'_{001} m'_{111} - m'_{100} m'_{012} - m'_{010} m'_{102}$$

$$+ m'^{2}_{001} m'_{110} + 2\, m'_{100} m'_{001} m'_{011} + 2\, m'_{010} m'_{001} m'_{101}$$

$$+ m'_{100} m'_{010} m'_{002} - 3\, m'_{100} m'_{010} m'^{2}_{001}$$

Relations between the k-statistics and the Moments about the Mean

By factoring equations A.52 and then substituting from equations A.53, one can determine the following relations between the k-statistics and the moments about the mean:

$$k_{110} = \left[\frac{n}{n-1}\right] m_{110} \qquad k_{200} = \left[\frac{n}{n-1}\right] m_{200} \qquad \text{(A.54)}$$

$$k_{101} = [\frac{n}{n-1}]\ m_{101} \qquad k_{020} = [\frac{n}{n-1}]\ m_{020}$$

$$k_{011} = [\frac{n}{n-1}]\ m_{011} \qquad k_{002} = [\frac{n}{n-1}]\ m_{002}$$

$$k_{111} = [\frac{n^2}{(n-1)(n-2)}]\ m_{111}$$

$$k_{300} = [\frac{n^2}{(n-1)(n-2)}]\ m_{300}$$

$$k_{030} = [\frac{n^2}{(n-1)(n-2)}]\ m_{030}$$

$$k_{003} = [\frac{n^2}{(n-1)(n-2)}]\ m_{003}$$

$$k_{210} = [\frac{n^2}{(n-1)(n-2)}]\ m_{210} \qquad k_{120} = [\frac{n^2}{(n-1)(n-2)}]\ m_{120}$$

(A.54 Cont.)

$$k_{201} = [\frac{n^2}{(n-1)(n-2)}]\ m_{201} \qquad k_{102} = [\frac{n^2}{(n-1)(n-2)}]\ m_{102}$$

$$k_{021} = [\frac{n^2}{(n-1)(n-2)}]\ m_{021} \qquad k_{012} = [\frac{n^2}{(n-1)(n-2)}]\ m_{012}$$

$$k_{400} = \frac{n^2}{(n-1)(n-2)(n-3)}[(n+1)\ m_{400}\ -3(n-1)\ m_{200}^2]$$

$$k_{040} = \frac{n^2}{(n-1)(n-2)(n-3)}[(n+1)\ m_{040}\ -3(n-1)\ m_{020}^2]$$

$$k_{004} = \frac{n^2}{(n-1)(n-2)(n-3)}[(n+1)\ m_{004}\ -3(n-1)\ m_{002}^2]$$

$$k_{310} = \frac{n^2}{(n-1)(n-2)(n-3)}[(n+1)\ m_{310}\ -3(n-1)\ m_{200}m_{110}]$$

$$k_{130} = \frac{n^2}{(n-1)(n-2)(n-3)} [(n+1)\ m_{130}\ -3(n-1)\ m_{110}m_{020}]$$

$$k_{301} = \frac{n^2}{(n-1)(n-2)(n-3)} [(n+1)\ m_{301}\ -3(n-1)\ m_{200}m_{101}]$$

$$k_{103} = \frac{n^2}{(n-1)(n-2)(n-3)} [(n+1)\ m_{103}\ -3(n-1)\ m_{101}m_{002}]$$

$$k_{031} = \frac{n^2}{(n-1)(n-2)(n-3)} [(n+1)\ m_{031}\ -3(n-1)\ m_{020}m_{011}]$$

$$k_{013} = \frac{n^2}{(n-1)(n-2)(n-3)} [(n+1)\ m_{013}\ -3(n-1)\ m_{011}m_{002}]$$

$$k_{220} = \frac{n^2}{(n-1)(n-2)(n-3)} [(n+1)\ m_{220}\ -(n-1)\ m_{200}m_{020}$$

$$-2(n-1)\ m_{110}^2]$$

(A.54 Cont.)

$$k_{202} = \frac{n^2}{(n-1)(n-2)(n-3)} [(n+1)\ m_{202}\ -(n-1)\ m_{200}m_{002}$$

$$-2(n-1)\ m_{101}^2]$$

$$k_{022} = \frac{n^2}{(n-1)(n-2)(n-3)} [(n+1)\ m_{022}\ -(n-1)\ m_{020}m_{002}$$

$$-2(n-1)\ m_{011}^2]$$

$$k_{211} = \frac{n^2}{(n-1)(n-2)(n-3)} [(n+1)\ m_{211}\ -(n-1)\ m_{200}m_{011}$$

$$-2(n-1)\ m_{110}m_{101}]$$

$$k_{121} = \frac{n^2}{(n-1)(n-2)(n-3)} \; [(n+1)\; m_{121} - (n-1)\; m_{020}m_{101}$$

$$-2(n-1)\; m_{110}m_{011}]$$

(A.54 Cont.)

$$k_{112} = \frac{n^2}{(n-1)(n-2)(n-3)} \; [(n+1)\; m_{112} - (n-1)\; m_{002}m_{110}$$

$$-2(n-1)\; m_{101}m_{011}]$$

Variance Formulas

The following formulas will be useful in determining standard errors:

$$\text{var}(m'_{r00}) = \frac{1}{N} (\mu'_{2r,0,0} - \mu'^2_{r,0,0}) \qquad \text{(A.55)}$$

$$\text{cov}(m'_{q00}, m'_{r00}) = \frac{1}{N} (\mu'_{q+r,00} - \mu'_{q00}\mu'_{r00}) \qquad \text{(A.56)}$$

$$\text{var}(m_{r00}) = \frac{1}{N} (\mu_{2r,0,0} - \mu^2_{r,0,0} + r^2\; \mu_{2,0,0}\mu^2_{r-1,0,0}$$

$$- 2r\mu_{r-1,0,0}\mu_{r+1,0,0}) \qquad \text{(A.57)}$$

$$\text{cov}(m_{q00}, m_{r00}) = \frac{1}{N} (\mu_{r+q,0,0} - \mu_{r00}\mu_{q00}$$

$$+ rq\mu_{2,0,0}\mu_{r-1,0,0}\mu_{q-1,0,0}$$

$$- r\mu_{r-1,0,0}\mu_{q+1,0,0}$$

$$- q\, \mu_{r+1,0,0}\, \mu_{q-1,0,0}) \qquad \text{(A.58)}$$

$$\mathrm{var}(m'_{r,s,0}) = \frac{1}{N} (\mu'_{2r,2s,0} - \mu'^{2}_{r,s,0}) \qquad \text{(A.59)}$$

$$\mathrm{cov}(m'_{r,s,0}, m'_{u,v,0}) = \frac{1}{N} (\mu'_{r+u,\ s+v,\ 0} - \mu'_{r,s,0}\mu'_{u,v,0}) \qquad \text{(A.60)}$$

$$\mathrm{var}(m_{r,s,0}) = \frac{1}{N} (\mu_{2r,2s,0} - \mu^2_{r,s,0} + r^2\, \mu_{2,0,0}\mu^2_{r-1,s,0}$$

$$+ s^2\, \mu_{0,2,0}\, \mu^2_{r,s-1,0}$$

$$+ 2\, rs\mu_{1,1,0}\mu_{r-1,s,0}\mu_{r,s-1,0}$$

$$- 2\, r\mu_{r+1,s,0}\, \mu_{r-1,s,0} - 2s\, \mu_{r,s+1,0}\, \mu_{r,s-1,0}) \qquad \text{(A.61)}$$

$$\mathrm{cov}(m_{r,s}, m_{u,v}) = \frac{1}{N} (\mu_{r+u,\ s+v,\ 0} - \mu_{r,s,0}\, \mu_{u,v,0}$$

$$+ ru\, \mu_{2,0,0}\, \mu_{r-1,s,0}\, \mu_{u-1,v,0}$$

$$+ sv\mu_{0,2,0}\, \mu_{r,s-1,0}\, \mu_{u,\ v-1,0}$$

$$+ rv\, \mu_{1,1,0}\mu_{r-1,s,0}\, \mu_{u,v-1,0}$$

$$+ su\mu_{1,1,0}\, \mu_{r,s-1,0}\, \mu_{u-1,v,0}$$

$$- u\, \mu_{r+1,s,0}\, \mu_{u-1,v,0}$$

$$- s\, \mu_{r,s-1,0}\, \mu_{u,v+1,0}$$

$$-v\, \mu_{r,\, s+1,\, 0}\, \mu_{u,\, v-1,\, 0}$$

$$-r\, \mu_{r-1,s,0}\, \mu_{u+1,v,0}) \qquad \text{(A.62)}$$

$$\mathrm{var}\ (\gamma_1) = \frac{6\ N(N-1)}{(N-2)(N+1)(N+3)} = \frac{\mathrm{var}(\beta_1)}{4\ \gamma_1^{\,2}} \qquad \text{(A.63)}$$

$$\mathrm{var}\ (\gamma_2) = \frac{24\ N(N-1)^2}{(N-3)(N-2)(N+3)(N+5)} = \mathrm{var}(\beta_2) \qquad \text{(A.64)}$$

$$\mathrm{var}\ k_{200} = \frac{1}{N}\,\kappa_{400} + \frac{2}{N-1}\,\kappa_{200}^{\;2} \qquad \text{(A.65)}$$

$$\mathrm{var}\ k_{020} = \frac{1}{N}\,\kappa_{040} + \frac{2}{N-1}\,\kappa_{020}^{\;2} \qquad \text{(A.66)}$$

$$\mathrm{var}\ k_{002} = \frac{1}{N}\,\kappa_{004} + \frac{2}{N-1}\,\kappa_{002}^{\;2} \qquad \text{(A.67)}$$

$$\mathrm{var}\ k_{110} = \frac{1}{N}\,\kappa_{220} + \frac{1}{N-1}\,\kappa_{200}\kappa_{020} + \frac{1}{N-1}\,\kappa_{110}^{\;2} \qquad \text{(A.68)}$$

$$\mathrm{var}\ k_{101} = \frac{1}{N}\,\kappa_{202} + \frac{1}{N-1}\,\kappa_{200}\kappa_{002} + \frac{1}{N-1}\,\kappa_{101}^{\;2} \qquad \text{(A.69)}$$

$$\mathrm{var}\ k_{011} = \frac{1}{N}\,\kappa_{022} + \frac{1}{N-1}\,\kappa_{020}\kappa_{002} + \frac{1}{N-1}\,\kappa_{011}^{\;2} \qquad \text{(A.70)}$$

$$\mathrm{var}\ k_{300} = \frac{1}{N}\,\kappa_{600} + \frac{9}{N-1}\,\kappa_{400}\kappa_{200} + \frac{9}{N-1}\,\kappa_{300}^{2}$$

$$+ \frac{6N}{(N-1)(N-2)}\,\kappa_{200}^{3} \qquad \text{(A.71)}$$

$$\mathrm{var}\ k_{400} = \frac{1}{N}\,\kappa_{800} + \frac{16}{N-1}\,\kappa_{600}\kappa_{200} + \frac{48}{N-1}\,\kappa_{500}\kappa_{300}$$

$$+ \frac{34}{N-1} \kappa^2_{400} + \frac{72N}{(N-1)(N-2)} \kappa_{400} \kappa^2_{200}$$

$$+ \frac{144 N}{(N-1)(N-2)} \kappa^2_{300}\kappa_{200} + \frac{24N(N-1)}{(N-1)(N-2)(N-3)} \kappa^4_{200} \qquad (A.72)$$

Now since $E(k_p) = \kappa_p$, to a first approximation the κ's in the above formulas can be replaced with the corresponding k's. Now

$$E(k^2_{200}) \neq \kappa^2_{200} \text{ rather } E(\ell_{22}) = \kappa^2_{200}$$

where $\ell_{22} = \frac{N-1}{N+1} (k^2_{200} - \frac{k_{400}}{N})$. This involves a correction factor of order $\frac{1}{N}$ to the method of replacing the κ's with the k's. For the present work this is an unnecessarily fine correction. However if desired the univariate formulas for the ℓ-statistics called "poly-kays" can be found in Wishart [1952].

Now consider the formulas for the variances of functions of random variables. If $\xi_1\xi_2,\xi_3 \ldots \xi_n$ are random variables with corresponding variates $x_1,x_2,x_3 \ldots x_n$, one may define a function $g(\xi_1,\xi_2, \ldots \xi_n)$ as that function which takes the value $g(x_1,x_2, \ldots x_n)$ when

$$\xi_1 = x_1,\ \xi_2, = x_2 \ldots \xi_n = x_n$$

Suppose that x_i has mean θ_i and $\frac{\partial q}{\partial \theta_i} = \frac{\partial g}{\partial x_i}\Big|_{x_i = \theta_i}$

Then

$$\text{var } g = \sum_{i,j=1}^{n} \left\{\frac{\partial g}{\partial \theta_i} \frac{\partial g}{\partial \theta_j} \text{cov}(x_i,x_j)\right\} \quad \text{(A.73)}$$

also for two functions of random variables

$$\text{cov}(g,h) = \sum_{i,j=1}^{n} \left\{\frac{\partial g}{\partial \theta_i} \frac{\partial h}{\partial \theta_j} \text{cov}(x_i,x_j)\right\} \quad \text{(A.74)}$$

Three cases are of interest:

(I) g is a function of one variable

$$\text{var } g(x) = \left(\frac{dg}{d\theta}\right)^2 \text{var } x \quad \text{(A.75)}$$

(II) g is a linear function $g = \sum a_i x_i$

$$\text{var } g = \sum a_i^2 \text{ var } x_i + \sum_{i \neq j} a_i a_j \text{ cov}(x_i,x_j) \quad \text{(A.76)}$$

(III) g is a ratio x_1/x_2 $\quad x_2 \geq 0$

$$\text{var}(x_1/x_2) = \frac{\text{var } x_1}{\theta_2^2} + \frac{\theta_1^2 \text{ var } x_2}{\theta_2^4} - \frac{2\,\theta_1 \text{ cov}(x_1,x_2)}{\theta_2^3} \quad \text{(A.77)}$$

$$\text{or var}(x_1/x_2) = \left\{\frac{E(x_1)}{E(x_2)}\right\}^2 \left\{\frac{\text{var } x_1}{E^2(x_1)} + \frac{\text{var } x_2}{E^2(x_2)} - \frac{2 \text{ cov } (x_1,x_2)}{E(x_1)\,E(x_2)}\right\} \quad \text{(A.78)}$$

Now $r = m_{11}/(m_{20}m_{02})^{1/2}$ and $\rho = \mu_{11}/(\mu_{20}\mu_{02})^{1/2}$

$$\mathrm{var}\ (r) = \frac{\rho^2}{N}\ \left\{\frac{\mu_{22}}{\mu_{11}^2} + \frac{1}{4}\left(\frac{\mu_{40}}{\mu_{20}^2} + \frac{\mu_{04}}{\mu_{02}^2} + \frac{2\ \mu_{22}}{\mu_{20}\mu_{02}}\right)\right.$$

$$\left. -\left(\frac{\mu_{31}}{\mu_{11}\mu_{20}} + \frac{\mu_{13}}{\mu_{11}\mu_{02}}\right)\right\} \tag{A.79}$$

<u>The Gaussian Distribution</u>

The general form

$$dF = \frac{1}{\sigma(2\pi)^{1/2}}\ \exp\left\{-\frac{1}{2\sigma^2}\ (x-\mu_1')^2\right\}dx, \tag{A.80}$$

$$\text{where}\quad -\infty \le x \le \infty \quad \text{and} \quad \sigma > 0$$

is known as the univariate Gaussian or normal distribution. The characteristic function is given by

$$C(t) = \exp(it\mu_1' - \frac{1}{2}\ t^2\ \sigma^2) \tag{A.81}$$

So

$$\mu_{2r} = \frac{(2r)!}{2^r\ r!}\ \sigma^{2r} \tag{A.82}$$

and $\qquad \mu_{2r+1} = 0 \quad \text{for } r \ge 1$ (A.83)

or in terms of cumulants $\quad \kappa_2 = \sigma^2$ (A.84)

$$\kappa_r = 0 \qquad \text{for } r > 2. \tag{A.85}$$

For this distribution

$$\beta_1 = 0 \quad , \quad \gamma_1 = 0$$

$$\beta_2 = 3 \quad , \quad \gamma_2 = 0$$

Thus the Gaussian distribution is a mesokurtic distribution.

The form of the trivariate Gaussian distribution is

$$dF = (2\pi)^{-3/2} \ |\overleftrightarrow{A}|^{1/2} \exp \{-\tfrac{1}{2} (\underset{\sim}{v} - \underset{\sim}{\mu})^{\dagger} \ \overleftrightarrow{A} \ (\underset{\sim}{v}-\underset{\sim}{\mu})\} dv_1 dv_2 dv_3 \tag{A.86}$$

where $\sim$ denotes a column vector, a $\dagger$ denotes transposition into a row vector, and $\leftrightarrow$ denotes a matrix. In this case $\overleftrightarrow{A}$ is a symmetric (3×3) matrix which is the inverse of the matrix of variances and covariances. The variates are v_1,v_2, and v_3, and

$\underset{\sim}{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$. Now μ is defined as $\begin{bmatrix} \mu'_{100} \\ \mu'_{010} \\ \mu'_{001} \end{bmatrix}$. The inverse of $\overleftrightarrow{A}$ is $\overleftrightarrow{A}^{-1}$

(the matrix of variances and covariances) and is expressed by

$$\overleftrightarrow{A}^{-1} = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \rho_{13}\sigma_1\sigma_3 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & \rho_{23}\sigma_2\sigma_3 \\ \rho_{13}\sigma_1\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \sigma_3^2 \end{bmatrix} \tag{A.87}$$

$$\underset{\longleftrightarrow}{A} = \begin{bmatrix} \frac{(1-\rho_{23}^2)}{K\sigma_1^{\,2}} & -\frac{(\rho_{12}-\rho_{13}\rho_{23})}{K\sigma_1\sigma_2} & -\frac{(\rho_{13}-\rho_{12}\rho_{23})}{K\sigma_1\sigma_3} \\ -\frac{(\rho_{12}-\rho_{13}\rho_{23})}{K\sigma_1\sigma_2} & \frac{(1-\rho_{13}^2)}{K\sigma_2^{\,2}} & -\frac{(\rho_{23}-\rho_{12}\rho_{13})}{K\sigma_2\sigma_3} \\ -\frac{(\rho_{13}-\rho_{12}\rho_{23})}{K\sigma_1\sigma_3} & -\frac{(\rho_{23}-\rho_{12}\rho_{13})}{K\sigma_2\sigma_3} & \frac{(1-\rho_{12}^2)}{K\,\sigma_3^2} \end{bmatrix} \tag{A.88}$$

$$\text{where } K=1-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2+2\,\rho_{12}\rho_{13}\rho_{23} \tag{A.89}$$

The determinate of $\underset{\longleftrightarrow}{A}$ is

$$|\underset{\longleftrightarrow}{A}| = \frac{1}{K\ \sigma_1^{\,2}\sigma_2^{\,2}\sigma_3^{\,2}} \tag{A.90}$$

So

$$|\underset{\longleftrightarrow}{A}^{-1}| = K\ \sigma_1^{\,2}\sigma_2^{\,2}\sigma_3^{\,2} \tag{A.91}$$

Now by definition the turbulent velocities have zero for the mean value, so $\mu'_{100} = \mu'_{010} = \mu'_{001} = 0$. Thus with this assumption, when equation A.86 is written out in long form it becomes

$$dF = (2\pi)^{-3/2}\ (K\sigma_1^{\,2}\sigma_2^{\,2}\sigma_3^{\,2})^{-1/2} \quad \exp\Big\{-\frac{1}{2K}\ \Big[\frac{(1-\rho_{23}^2)}{\sigma_1^2}\ v_1^{\,2}$$

$$- \frac{2(\rho_{12}-\rho_{13}\rho_{23})}{\sigma_1\sigma_2} v_1v_2 - \frac{2(\rho_{13}-\rho_{12}\rho_{23})}{\sigma_1\sigma_3} v_1v_3 + \frac{(1-\rho_{13}^2)}{\sigma_2^2} v_2^2$$

$$- \frac{2(\rho_{23}-\rho_{12}\rho_{13})}{\sigma_2\sigma_3} v_2v_3 + \frac{(1-\rho_{12}^2)}{\sigma_3^2} v_3^2]\}dv_1dv_2dv_3 \tag{A.93}$$

The characteristic function is

$$C(t_1,t_2,t_3) = \exp\{-\frac{1}{2}(\sigma_1^2t_1^2 + 2\rho_{12}\sigma_1\sigma_2t_1t_2 + 2\rho_{13}\sigma_1\sigma_3t_1t_3$$

$$+ \sigma_2^2t_2^2 + 2\rho_{23}\sigma_2\sigma_3t_2t_3 + \sigma_3^2t_3^2)\} \tag{A.94}$$

The Chi-Square Distribution

If $x_1,x_2 \ldots x_N$ are N independent normal variates with means $\mu_1, \mu_2 \ldots \mu_N$ and variances $\sigma_1, \sigma_2 \ldots \sigma_N$, then the chi-square variate is given by

$$\chi^2 = \sum_{i=1}^{N} \left(\frac{x_i - \mu_i}{\sigma_i}\right)^2 \tag{A.95}$$

The chi-square distribution is

$$dF(\chi^2) = \left(\frac{\chi^2}{2}\right)^{(\frac{N}{2})-1} \frac{e^{-\chi^2/2}}{2\Gamma(N/2)} d(\chi^2) \tag{A.96}$$

where $0 \leq \chi^2 \leq \infty$, and N is called the number of degrees of freedom, and Γ is the gamma function. The r th cumulant for χ^2 becomes

$$\kappa_r = 2^{r-1} (r-1)!\, N \qquad \text{(A.97)}$$

The following are some properties of the chi-square distribution: If Y_i, where $i=1,2,\ldots n$, is one of a set of orthogonal linear functions of the independent variates $x_j (j=1,2,\ldots n)$, and if the x_j are standardized variates (they have mean zero and variance one), then the distribution of $\sum_i Y_i^2$ is a chi-square distribution with n degrees of freedom; the sum of two independent chi-square variates with n_1 and n_2 degrees of freedom is a chi-square variate with $n_1 + n_2$ degrees of freedom; if

$A = \sum_{j=1}^{n} x_j^2$ and $B = \sum_{i=1}^{h} Y_i^2$, where the Y_i are orthogonal linear functions of the independent standard normal variates x_j, then A-B is a chi-square variate with n-h degrees of freedom, and is independent of B.

"Student's" t-distribution

The distribution function of t is given by

$$dF = \left\{ n^{1/2} \frac{\Gamma(1/2)\Gamma(n/2)}{\Gamma(\frac{n+1}{2})} (1 + \frac{t^2}{n})^{(n+1)/2} \right\}^{-1} dt \qquad \text{(A.98)}$$

where $n > 0$ and $-\infty \leq t \leq \infty$.

The variate is t, and it is

$$t = N^{1/2} \frac{(\langle x \rangle - \mu)}{s} \qquad \text{(A.99)}$$

where N is the number of observations in the sample, μ is the population mean, $\langle x \rangle$ is the sample mean, and s is the square root of the sample variance. In this formula, n=N-1 and is called the number of degrees of freedom for the distribution. An important characteristic of this distribution is that it is independent of the population variance.

The moments μ_r' of the distribution exist only for $r < n$; odd-order moments are equal to zero by symmetry and even moments can be shown to be

$$\mu_{2r}' = n^r \frac{\Gamma(r + 1/2)\ \Gamma(n/2 - r)}{\Gamma(1/2)\ \Gamma(n/2)}\ ,\ 2r < n. \tag{A.100}$$

"Fisher's" F and z distributions

Let Q_1^2 and Q_2^2 be the observed variances for two samples, of sizes N_1 and N_2 respectively, drawn from normal populations with equal variances. The sample variance ratio is

$$F = \frac{Q_1^2}{Q_2^2} = \frac{\sum_{i=1}^{N_1} (x_{1_i} - \langle x_1 \rangle)^2/(N_1-1)}{\sum_{j=1}^{N_2} (x_{2_i} - \langle x_2 \rangle)^2/(N_2-1)} \tag{A.101}$$

The sampling distribution of the sample variance ratio is

$$dG = \frac{\left(\frac{n_1}{n_2}\right)^{n_1/2} \Gamma\left(\frac{n_1 + n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right) \Gamma\left(\frac{n_2}{2}\right)} \cdot \frac{F^{\frac{n_1 - 2}{2}}}{\left(1 + \frac{n_1 F}{n_2}\right)^{(n_1 + n_2)/2}} \, dF \qquad \text{(A.102)}$$

where $n_1 = N_1 - 1$ and $n_2 = N_2 - 1$, $n_1, n_2 > 0$, and $0 \leq F \leq \infty$.

Hence n_1 and n_2 are the degrees of freedom for the samples. Fisher originally used the related statistic

$$z = \frac{1}{2} \log F \qquad \text{(A.103)}$$

whose distribution is

$$dH = \frac{2\, n_1^{(n_1)/2}\, n_2^{(n_2)/2}\, \Gamma\left(\frac{n_1 + n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right) \Gamma\left(\frac{n_2}{2}\right)} \cdot \frac{e^{n_1 z}}{(n_1 e^{2z} + n_2)^{(n_1 + n_2)/2}} \, dz \qquad \text{(A.104)}$$

where n_1, $n_2 > 0$ and $-\infty \leq z \leq +\infty$

Modern practice is almost entirely in favor of using the simpler statistic F.

Tables for the distribution function for the univariate Gaussian distribution, the t distribution, the F distribution, and the χ^2 distribution have been developed; a bibliography of tables may be found in Kendall [1958].

Theorem I

If the joint distribution of the three variates (v_1, v_2 and v_3) is a trivariate normal or Gaussian distribution, then the marginal distribution of each one of the three variates is a univariate normal or Gaussian distribution with the variances unchanged.

Now the contra-positive statement of the above theorem holds true in its own right, and this statement is theorem II.

Theorem II

If at least one of the three marignal distributions concerned with the variates (v_1,v_2, and v_3) is not a univariate normal or Gaussian distribution, then the joint distribution of the three variates is not a trivariate normal or Gaussian distribution.

Theorem III, the Central Limit Theorem

Let $x_1,x_2,\ldots x_N$ be independent random variates all having the same distribution with mean μ and variance σ^2, but not necessarily a normal distribution. Let the standardized variate corresponding to x_j be

$$z_j = \frac{x_j - \mu}{\sigma} \tag{A.105}$$

and let Y_N be defined by

$$Y_N = \frac{\sum\limits_{j=1}^{N} z_j}{N^{1/2}} = N^{1/2} < z > \tag{A.106}$$

where $< z >$ is the mean of the z_j. Then the theorem states that as

$N \to \infty$, Y_N tends to a standard normal variate. In other words, if $\alpha(t)$ represents the p.d.f. for the normal distribution then

$$F(Y_N) = \int_{-\infty}^{Y_N} \alpha(t)\,dt \qquad \text{(A.107)}$$

The point of the theorem is that, no matter what the original distribution of z_j may be, the mean of a large enough sample will have a nearly normal distribution.

Statistical Tests

General Catagories

There are many catagories which one can use to classify the many different statistical tests. However, there are three catagories which are particularly relevant to the statistical tests used in this work. The three are the following:

(1) Tests of the significance of a specified parameter value. The typical hypothesis here is that a parameter in a population of known form has a specified value. The question to be answered is whether or not the evidence provided by the sample supports the hypothesis.

(2) Tests of goodness of fit. The hypothesis is that the population is of a certain kind which is either fully specified beforehand or can be "estimated" with the help of the data. One wishes to know if the sample values fit this population in the sense that they could have arisen from it (with any acceptable degree of probability) by random sampling. This hypothesis is more general than that of

(1) because it concerns the whole distribution function and not merely one of its parameters.

(3) Tests of homogeneity. The hypothesis here concerns two or more populations, each providing a contribution to the sample. One wishes to test whether the populations have certain parameters in common, or in the extreme case, whether they are identical. In the case of dissimilar populations, by proper partitioning of the sample one can sometimes determine the factors which produce the dissimilarity of the parameters and thereby necessitate the separation into different populations.

Discussion about Testing

The usual procedure is to set up a hypothesis and then test it by experiment. The hypothesis is called a null hypothesis. In the present work the experimentation was completed before the hypothesis was tested. On the basis of the sample one can take various possible actions. One can (1) reject the hypothesis, (2) accept the hypothesis, or (3) declare that further experimentation is necessary before one can make a decision.

In taking an action on the basis of a sample one runs a risk of doing the wrong thing. Obviously one of two kinds of errors may be committed; the rejection of a hypothesis which is really true (this will be called a rejection error or an error of the first kind), or the acceptance of a hypothesis which is really false (this will be called an acceptance error or an error of the second kind).

Tests are usually made by computing some statistic (e.g., a mean or a variance) from the observations and finding whether or not this computed value lies within some particular interval, or set of intervals, previously chosen of the axis of real numbers. The part of the real axis so chosen is called the region of rejection, and the null hypothesis H_0 is rejected if the computed value lies in this region. For example, if the population is known to be normally distributed about a mean value μ with unit variance, and if H_0 is the hypothesis that μ is zero, one would be inclined to reject the hypothesis H_0 if a sample of N observations gives a mean too distant from zero. If H_0 were true, the sample mean would depart from zero by as much as $1.96\ N^{-1/2}$ in only five percent of random samples of size N. By taking as the region of rejection that part of the real axis outside the bounds $\pm\ 1.96\ N^{-1/2}$, one runs a risk of wrongly rejecting H_0, but the chance of doing so is only 0.05. By suitably choosing the region of rejection one can make this chance as small as is necessary, depending upon the circumstances of the problem and the consequences of making a wrong decision.

Let the theoretical value be θ and the estimate from a sample be t. Statisticians in general accept the following convention: if the probability of obtaining by chance a sample with a difference numerically as great as θ-t is less than 0.05, the observed difference is significant; if the probability is less than 0.01, the difference is highly significant; if the probability is greater than 0.05, the difference is non-significant.

An hypothesis which is equivalent to a complete specification of the distribution is said to be a simple hypothesis, otherwise it is a composite hypothesis. Thus, if a population is known to be normal and to have variance σ^2, the hypothesis that the mean is μ_0 is a simple hypothesis, since the mean and the variance together specify a normal distribution completely. The alternative hypothesis could be simple, but usually it is composite. An example of a two-sided alternative hypothesis, which is composite, is that μ is either less than or greater than μ_0.

The Size and Power of a Test

Consider the testing of the simple null hypothesis (H_0), that $\theta = \theta_0$, against the simple alternative hypothesis (H_1), that $\theta = \theta_1$, by means of a test statistic T. In order to fix the region of rejection, the probability density function of T when $\theta = \theta_0$ must be known. This p.d.f. is denoted by $f(t,\theta_0)$. In general, f will depend on the sample size N, and the region of rejection (R) will be an interval on the t axis (for example, the interval $t > t_\alpha$ in Figure 29).

If the probability that T falls in R (when H_0 is true) is α, then α is the probability of committing a rejection error. This probability is called the size of the test. It is given by

$$\alpha = \int_{(R)} f(t,\theta_0)dt \qquad \text{(A.108)}$$

and is represented by the heavily-shaded area in Figure 29. In practice, R is chosen so that α is 0.05 or 0.01.

If the alternative hypothesis H_1 is true, there will be a different distribution of T, with the new p.d.f. $f(t,\theta_1)$. The probability of committing an acceptance error (that is of accepting H_0 when H_1 is true) is given by

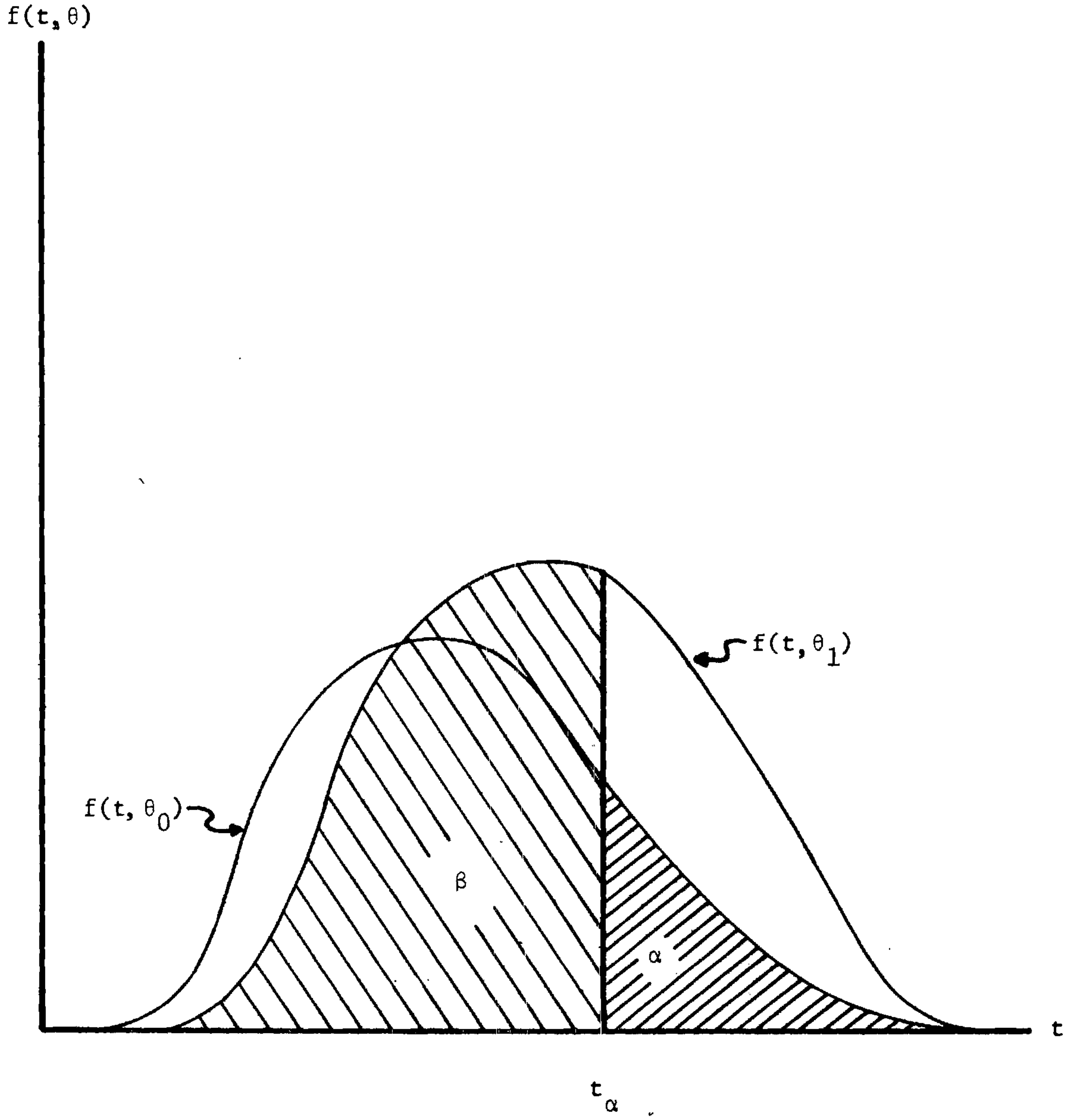

Figure 29. The Rejection Error and the Acceptance Error.

$$\beta = \int_{(A)} f(t,\theta_1)dt \tag{A.109}$$

where A is the region of acceptance (all possible values of t outside of R). This probability is represented by the lightly-shaded area in Figure 29. The power of the test is defined by

$$\text{Power} = 1 - \beta = \int_{(R)} f(t,\theta_1)dt \tag{A.110}$$

The power of the test is the probability of rejecting H_0 if H_1 is true. Obviously, the test should be as powerful as possible for the same size.

The Likelihood Ratio Test.

Let the possible values of θ form a set Ω (which may, for example, be the interval 0 to 1, or the interval $-\infty$ to ∞). The null hypothesis H_0 specifies that θ belongs to some subset w of Ω (for example, the single value 0.5, or the interval 0.4 to 0.6) and H_1 is then the hypothesis that θ belongs to Ω - w. The likelihood ratio is defined as the ratio of the maximum likelihood under H_0 to the maximum likelihood under H_1:

$$L(x) = \frac{\max\limits_{\theta \in w} f(x,\theta)}{\max\limits_{\theta \in \Omega - w} f(x,\theta)} \tag{A.111}$$

The likelihood ratio is denoted by L(x). Here f(x,θ) is the p.d.f. for x given that θ is in some interval. If L(x) is small, the observed x will be more likely H_1 than under H_0, so that it would be unreasonable to maintain H_0. The test consists in rejecting H_0 when $L(x)<c$, c being such that the probability that $L(x)<c$ (under the hypothesis H_0) is equal to α. Many useful tests in statistics are likelihood ratio tests. The statistic x may consist of a set of N independent observations, forming a random sample; in this case recall that the likelihood will be a joint probability density for the N observations.

The One-Sided Normal Test

Let the population be normal with known variance σ^2. Let the test attempt to distinguish between two simple hypotheses: H_0 where the mean $\mu = \mu_0$, and H_1 where the mean $\mu = \mu_1$. The test statistic is the sample mean m computed from N observations. Since the sampling distribution of m is a normal distribution with mean μ and variance σ^2/N, the p.d.f. of m for a mean value μ is given by

$$f(m,\mu) = \left(\frac{N}{2\pi\,\sigma^2}\right)^{1/2} \exp\left\{-\frac{N(m-\mu)^2}{2\sigma^2}\right\} \qquad (A.112)$$

The likelihood ratio is

$$L(m) = \frac{f(m,\mu_0)}{f(m,\mu_1)} = \exp\left\{\frac{N}{2\sigma^2}\left[(m-\mu_1)^2 - (m-\mu_0)^2\right]\right\}$$

$$= \exp\left\{\frac{N}{2\sigma^2}(\mu_1 + \mu_0 - 2m)(\mu_1 - \mu_0)\right\} \qquad (A.113)$$

This will be less than some positive constant c if $\mu_1 + \mu_0 - 2m < c_1$, where

$$c_1 = \log (c) \;/[\frac{N}{2\sigma^2} (\mu_1 - \mu_0)] \tag{A.114}$$

The relation $\mu_1 + \mu_0 - 2\,m < c_1$ implies that

$$m > c_2 \quad \text{where } c_2 = \frac{\mu_1 + \mu_0 - c_1}{2}$$

Thus the test reduces to rejecting H_0 if the sample mean is greater than a certain value c_2. This value can be determined by deciding on the size of the test.

If α is the probability that $m > c_2$, given that $\mu = \mu_0$, then

$$\alpha = \int_{c_2}^{\infty} f(m,\mu_0)\; dm \tag{A.115}$$

By the transformation $x = \dfrac{(m - \mu_0)N^{1/2}}{\sigma}$, this becomes

$$\alpha = (2\pi)^{-1/2} \int_{x_0}^{\infty} e^{-x^2/2}\; dx$$

$$\alpha = 1 - \phi(x_0) \tag{A.116}$$

where $x_0 = \dfrac{(C_2 - \mu_0)N^{1/2}}{\sigma}$ and $\phi(x)$ is the c.d.f. for the normal distribution. If one chooses $\alpha = 0.05$, one finds from the tables of the normal law that $x_0 = 1.645$ so that

$$C_2 = \mu_0 + 1.645\ \sigma\ N^{-1/2} \tag{A.117}$$

The test therefore consists in rejecting H_0 in favor of H_1 if $m > \mu_0 + 1.645\ \sigma\ N^{-1/2}$.

This test can also be used as a test of the simple hypothesis H_0 (that $\mu = \mu_0$) against the composite alternative H_1 (that $\mu > \mu_0$). The set Ω of possible values of μ is the set of real numbers $\geq \mu_0$, and the set ω consists of the single number μ_0. The likelihood ratio is

$$L(m) = \frac{f(m,\mu_0)}{\max\limits_{\mu > \mu_0} f(m,\mu)} \tag{A.118}$$

where $f(m,\mu)$ is given by equation A.112. This p.d.f. $f(m,\mu)$ is a maximum for variations in μ when the exponential factor is equal to 1, that is when $\mu = m$. Thus

$$L(m) = \exp\left\{-\frac{N}{2\sigma^2}(m - \mu_0)^2\right\} \tag{A.119}$$

this is less than c if $m - \mu_0 > c_1$ where

$$c_1 = - \log (c)/[\frac{N}{2\sigma^2}] \tag{A.120}$$

this implies $m > c_2$ where $c_2 = c_1 + \mu_0$. Again the number c_2 is determined by

$$\alpha = \int_{c_2}^{\infty} f(m,\mu_0)dm = 1 - \phi(x_0) \tag{A.121}$$

where $x_0 = \frac{N^{1/2}}{\sigma} (c_2 - \mu_0)$ and where ϕ is the c.d.f. of the normal distribution. The test consists in rejecting H_0 if $m > c_2$.

The t-Test

Let there exist a sample from a distribution which is known to be normal, but whose mean and variance are unknown. One wishes to test the significance of a given value μ_0 of the mean, that is to say, one wishes to consider whether the observations could, to any acceptable probability, have been derived from a distribution with mean μ_0, whatever the variance may be.

The procedure is to calculate the statistic, which is the ratio of the difference between the sample mean and the population mean to the positive square root of the variance

$$t = \frac{(\langle x\rangle - \mu_0)}{\sqrt{\mathrm{var}\ (\langle x\rangle)}} = N^{1/2} \frac{(\langle x\rangle - \mu_0)}{S} \tag{A.122}$$

where N is the number of observations in the sample, $\langle x\rangle$ is the sample mean, and S is the standard deviation of the sample. The distribution of t is given by equation A.98, and thus one can find the probability that the calculated value of t is attained or exceeded. If the probability is small one rejects μ_0; if not, one accepts it. In practice, one does not have to calculate this probability because tables of the probability for given values of t and for the degrees of freedom (equal to N-1) have been developed.

The F-test

Suppose that S_1^2 and S_2^2 are the observed variances for two samples, of sizes N_1 and N_2, which were drawn from normal populations with variances σ_1^2 and σ_2^2 respectively. One can test the null hypothesis H_0 that $\sigma_1^2 = \sigma_2^2$ by determining the distribution of the variance ratio

$$F = \frac{S_1^2}{S_2^2} \tag{A.123}$$

The distribution of F is given in equation A.102. Thus, knowing the distribution, one can calculate the probability of observing the particular F which was computed from the samples. As usual, if this probability is small, one rejects the null hypothesis in favor of the alternative H_1 (that $\sigma_1^2 \neq \sigma_2^2$). Rather than computing the probability, in practice one resorts to a table which gives, for various values of $n_1 = N_1 - 1$ and $n_2 = N_2 - 1$, the upper 5% probability and the upper 1% probability points of the distribution of F. In order to use some of the tables it is necessary to place the larger mean square over the smaller mean square in the variance ratio. If interpolation for n_1 or n_2 is necessary, it should be harmonic interpolation instead of linear interpolation.

The M-Test

Suppose that one has K samples ($K > 2$) and that for the i th sample, of size N_i, the observed values are x_{i_j}, $j = 1,2, \ldots N_i$, $i = 1,2, \ldots K$. One assumes that all the samples are independent and come from normal populations with means $\mu_1, \mu_2 \ldots \mu_K$. The null hypothesis H_0 is that $\sigma_1^2 = \sigma_2^2 = \ldots = \sigma_K^2$ ($= \sigma^2$ say). The alternative hypothesis H_1 is that these variances are not equal. Under H_0 the likelihood function is

$$L_0 = \frac{1}{(2\pi\ \sigma^2)^{N/2}} \exp\left[-\frac{1}{2} \sum_{i,j} \left(\frac{x_{i_j} - \mu_i}{\sigma}\right)^2\right] \qquad \text{(A.124)}$$

where $N = \sum N_i$

under H_1 the likelihood function is

$$L_1 = \frac{1}{(2\pi)^{N/2} \sigma_1^{N_1} \ldots \sigma_K^{N_K}} \exp\left[-\frac{1}{2} \sum_{i,j} \left(\frac{x_{i_j} - \mu_i}{\sigma_i}\right)^2\right] \qquad \text{(A.125)}$$

The joint maximum likelihood estimators of μ_i and σ_i under H_0 are

$$\hat{\mu}_i = \frac{1}{N_i} \sum_j x_{i_j} = \langle x_i \rangle \qquad \text{(A.126)}$$

$$\hat{\sigma}^2 = \frac{1}{N} \sum_{i,j} (x_{i_j} - \langle x_i \rangle)^2 = \sum_i \frac{S_i}{N} \qquad \text{(A.127)}$$

where

$$S_i = \sum_j (x_{i_j} - \langle x_i \rangle)^2 \qquad \text{(A.128)}$$

The sample variance is $Q_i = \dfrac{S_i}{n_i}$

where

$$n_i = N_i - 1$$

The joint maximum likelihood estimators of μ_i and σ_i under H_1 are

$$\hat{\mu}_i = \langle x_i \rangle \qquad \text{(A.129)}$$

$$\hat{\sigma}_i^2 = \frac{S_i}{N_i} \qquad \text{(A.130)}$$

The likelihood ratio is

$$\frac{(L_0) \max}{(L_1) \max} = \frac{(S_1/N_1)^{N_1/2} \ldots (S_K/N_K)^{N_K/2}}{(S/N)^{N/2}} = L \qquad \text{(A.131)}$$

where $S = \sum S_i$ then H_0 will be rejected if $L < c$. The constant c is so chosen that the probability that $L < c$, given the hypothesis H_0, is less than or equal to α. Now the distribution of $-2 \log L$ for large N is approximately χ^2 with K-1 degrees of freedom. This number is the number of parameters under H_1, namely, 2K, less the number under H_0, namely, K + 1.

So

$$-2 \text{ Log } L = N \log \frac{S}{N} - \sum_i N_i \log \frac{S_i}{N_i} \tag{A.132}$$

The approximation to χ^2 may be improved by replacing each N_i by $n_i = N_i - 1$ and N by $n = N-k$. In effect this replaces the maximum likelihood estimators by unbiased estimators.

Thus the quantity $M = c^{-1} [n \log \frac{S}{n} - \sum_i n_i \log \frac{S_i}{n_i}]$ (A.133)

is distributed like $\chi 2$ with K-1 degrees of freedom where

$$c = 1 + \frac{\sum_i \frac{1}{n_i} - \frac{1}{n}}{3(K - 1)} \tag{A.134}$$

A small value of L will mean a large value of M, and this will lead to rejection of the null hypothesis. The procedure is to compute the value for M and then consult the table of χ^2 which gives for various degrees of freedom several values of M and the corresponding probabilities that these values could have arisen by chance. If the probability is small, H_0 will be rejected.

The χ^2 test

Suppose that the members of a population can all be placed in one or another of a set of k catagories. These catagories may be intervals of the domain of some measured variable. Also suppose that, according to a certain hypothesis H_0, the probabilities of falling in these classes should be π_1, π_2, ... π_k. This hypothesis may be tested by observing the actual frequencies f_1, f_2, ... f_k, in a sample of N items, corresponding to the respective classes. The distribution of N sample items among the k classes is multinomial, and the number of degrees of freedom is k - 1. The quantity

$$\chi_s^2 = \sum_{i=1}^{k} \frac{(f_i - N\pi_i)^2}{N\pi_i} \qquad \text{(A.135)}$$

has in the limit the χ^2 distribution with k-1 degrees of freedom. It is assumed in the above that the quantities π_i are given by the hypothesis H_0 and are not estimated from the sample. Usually, however, the parameters of the population will not be known from prior considerations but will have to be estimated from the sample. In this case, if s parameters are estimated by the method of maximum likelihood, the limiting distribution of χ_s^2 is χ^2 with k-s-1 degrees of freedom. Each additional parameter estimated from the sample reduces the degrees of freedom by one. The procedure is to compute the value χ_s^2 and then by consulting the table for χ^2 determine the probability that this value could have arisen by chance. If the probability is small, H_0 will be rejected.

Effect of Non-normality on Tests.

Since one does not usually know whether a sample comes from a normal population or not, it is natural to ask what difference it would make in the test if the population were not normal. Empirical data on sampling from artificial non-normal populations have shown that the t-test and the F-test (consequently the related M-test) give quite good results even for considerable departures from normality. The effect of non-normality is to slightly increase the apparent significance of results, which suggests caution when interpreting results very near the borderline of significance. The chi-square test is a non-parametric test and thus the validity of the test does not depend upon the assumption of normality for the population. In other words the chi-square test would not be affected by the non-normality.

APPENDIX B

ERROR IN TURBULENT VELOCITIES

A random sample of the globules used to determine the turbulent winds was selected. The method of sampling was as follows:

For each rocket the total number of observable globules was counted. The sample size for this rocket was then 1/10 of the total number. Each globule had previously been assigned an identification number. In order to decide which globules would enter the sample, the table of random numbers in Hodgeman [1959] was consulted. There were nine rocket releases involved and each release was assigned a column in the table. Progressing line by line in a column, the second and third digits from the right in the random number were examined. If these two digits corresponded to the identification number of a globule, then that globule was chosen for the sample and removed from further consideration; if the digits did not correspond to a globule then the number in the next line was considered. The process continued until the required number of globules for the sample of that rocket was obtained. Then the same procedure was initiated for the next rocket using its corresponding new column in the table.

Once a globule was selected, its already determined position errors (probable errors) were sampled throughout its lifetime. Hence several position errors were obtained for each globule. This was done due to the fact that the globule becomes larger, more diffuse, and more indistinct as it nears the end of its lifetime and its center-point is more difficult to determine and is subject to a larger error. Thus by sampling throughout its lifetime the error data is not biased either toward smaller errors near its beginning or toward larger errors at its end.

Next a frequency chart of the number of errors versus the size of the probable error was constructed, and from this chart an average value for the probable error in position was determined. This was done for errors in the N-S, the E-W, and the vertical directions.

Since the position of the globules was determined at 30 second intervals, the average value of the probable error divided by the time duration gave the average value of the probable error in the winds.

The standard error in the winds was assigned the value of 0.6745 divided into the average value of the probable error in the winds. The results are shown in Table 15.

Table 15. Errors in Turbulent Velocity Determinations

Type of Error	Error in Northward Velocity Component	Error in Eastward Velocity Component	Error in Vertical Velocity Component	Average Error of the Three Components
Probable Error in Winds, m/sec	1.94	1.62	2.16	1.91
Standard Error in Winds, m/sec	2.87	2.40	3.20	2.83

APPENDIX C

SOME USEFUL INTEGRALS

$$\int_{-\infty}^{\infty} e^{-q^2x^2} \, dx = \frac{\pi^{1/2}}{q} \tag{C.1}$$

$$\int_{-\infty}^{\infty} x^{2n} \, e^{-px^2} \, dx = \frac{(2n - 1)!!}{(2p)^n} \left(\frac{\pi}{p}\right)^{1/2} \tag{C.2}$$

where (5)!! = 5·3·1 = 15

$$\int_{-\infty}^{\infty} e^{-p^2x^2 \pm qx} \, dx = \frac{\pi^{1/2}}{p} \quad \exp\{q^2/4p^2\} \tag{C.3}$$

$$\int_{-\infty}^{\infty} x^n \, e^{-px^2 + 2qx} \, dx = \frac{1}{2^{n-1}p} \left(\frac{\pi}{p}\right)^{1/2} \frac{d^{n-1}}{dq^{n-1}} \left(qe^{\frac{q^2}{p}}\right) \tag{C.4}$$

APPENDIX D

PROOF THAT THE NON-GAUSSIAN CHARACTERISTIC OF A HIGH β_2 COULD NOT HAVE ARISEN FROM THE SIMULTANEOUS SAMPLING OF GAUSSIAN ERRORS AND GAUSSIAN TURBULENT VELOCITIES

With the assumptions that the turbulent velocity p.d.f. is a Gaussian p.d.f. with mean zero and variance σ_1^2 and that the error (in the observed velocities) p.d.f. is Gaussian with mean zero and variance σ_2^2, then the turbulent velocity p.d.f. $f_1(v)$ is given by

$$f_1(v) = \frac{e^{-\frac{v^2}{2\sigma_1^2}}}{\sqrt{2\pi}\ \sigma_1} \tag{D.1}$$

and the error p.d.f. $f_2(v)$ is

$$f_2(v) = \frac{e^{-\frac{v^2}{2\sigma_2^2}}}{\sqrt{2\pi}\ \sigma_2} \tag{D.2}$$

Since the two variates, the turbulent velocity and the error in the turbulent velocity, are statistically independent, the p.d.f. for the observed velocity v which simultaneously consists of a turbulent velocity v' and an error v'' (where $v'' = v - v'$) is

$$f(v) = \int_{\infty}^{\infty} f_1(v') \; f_2(v - v') \; dv' \tag{D.3}$$

which becomes

$$= \frac{1}{2\pi\sigma_1\sigma_2} \int_{\infty}^{\infty} \left\{\exp\left[-\frac{v'^2}{2\sigma_1^2}\right]\right\} \left\{\exp\left[-\frac{(v-v')^2}{2\sigma_2^2}\right]\right\} dv' \quad .$$

After factoring one has

$$= \frac{\exp\left[-\frac{v^2}{2\sigma_2^2}\right]}{2\pi \quad \sigma_1\sigma_2} \int_{\infty}^{\infty} \exp\left[-\frac{(\sigma_1^2 + \sigma_2^2)}{2\sigma_1^2 \; \sigma_2^2} v'^2 + \frac{vv'}{\sigma_2^2}\right] dv' \quad .$$

By equation C.3 one obtains

$$= \frac{\exp\left[-\frac{v^2}{2\sigma_2^2}\right]}{2\pi \; \sigma_1\sigma_2} \left[\left(\frac{2\pi\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}\right)^{1/2} \quad \exp\left\{\frac{\sigma_1^2 \; v^2}{2\sigma_2^2(\sigma_1^2 + \sigma_2^2)}\right\} \right],$$

which becomes

$$= [2\pi(\sigma_1^2 + \sigma_2^2)]^{-1/2} \quad \exp\left[-\frac{v^2}{2\sigma_2^2} + \frac{\sigma_1^2 \; v^2}{2\sigma_2^2(\sigma_1^2 + \sigma_2^2)}\right]$$

and upon rearranging one has

$$f(v) = [2\pi(\sigma_1^2 + \sigma_2^2)]^{-1/2} \exp[-\frac{v^2}{2(\sigma_1^2 + \sigma_2^2)}] \qquad (D.4)$$

which is a Gaussian p.d.f. with mean zero, variance $\sigma_1^2 + \sigma_2^2$, and $\beta_2 = 3.0$. Thus the observed velocity p.d.f. f(v) would be Gaussian, under the assumption of simultaneous sampling from Gaussian turbulent velocities and Gaussian errors. Since the actual observed velocity p.d.f. was not of Gaussian form, the above assumption is proven false.

APPENDIX E

DEMONSTRATION OF IMPROBABILITY THAT NON-GAUSSIAN CHARACTER IS DUE TO INTERMITTANT OBSERVATION OF GAUSSIAN TRUBULENT VELOCITIES PLUS GAUSSIAN ERROR OR ALTERNATIVELY OF GAUSSIAN ERROR ALONE

Assume that the experimenter is unaware of the fact that his set of sample observations is comprised of two distinct groups. The first group of observations of the velocity v is such that v consists of the turbulent velocity v' and an error v'' $(v = v' + v'')$. The second group of observations is such that v consists of only an error v'' $(v = v'')$.

With the further assumptions that the turbulent velocity p.d.f. is Gaussian with mean zero and variance $\sigma_1^{\,2}$ and that the error p.d.f. is Gaussian with mean zero and variance $\sigma_2^{\,2}$, then the error p.d.f. is given by equation D.2 and, as was shown in Appendix D, the p.d.f. for the first group of observations is given by

$$f_3(v) = [2\pi\,\sigma_3^{\,2}]^{-1/2} \exp[-\frac{v^2}{2\sigma_3^{\,2}}] \qquad \text{(E.1)}$$

where $\quad \sigma_3^{\,2} = \sigma_1^{\,2} + \sigma_2^{\,2}.$

Now, being unaware of the distinction between the two groups of observations, the experimenter would combine all observations into a single sample and attempt to draw conclusions about the non-Gaussian character of the turbulent velocity p.d.f. from the single sample. If, on the other hand, the intermittancy factor γ(which is the ratio of the number of observations in the second group to the total number of observations in both groups) were known, he should use the p.d.f. f(v) to represent the sample; where f(v) is given by

$$f(v) = \gamma f_2(v) + (1 - \gamma)\, f_3(v) \tag{E.2}$$

rewriting gives

$$f(v) = \gamma(2\pi\sigma_2^{\,2})^{-1/2} \exp[-\frac{v^2}{2\sigma_2^{\,2}}] + (1-\gamma)(2\pi\sigma_3^{\,2})^{-1/2} \exp[-\frac{v^2}{2\sigma_3^{\,2}}] \tag{E.3}$$

Now

$$\mu_2 = \int_{-\infty}^{\infty} v^2\, f(v)\,dv \tag{E.4}$$

by equation C.2 one obtains

$$\mu_2 = \gamma\sigma_2^{\,2} + (1-\gamma)\sigma_3^{\,2} \quad , \tag{E.5}$$

and

$$\mu_4 = \int_{-\infty}^{\infty} v^4\, f(v)\,dv \tag{E.6}$$

again by equation C.2 one obtains

$$\mu_4 = 3[\gamma\sigma_2^{\,4} + (1-\gamma)\,\sigma_3^{\,4}] \tag{E.7}$$

So

$$\beta_2 = \frac{\mu_4}{\mu_2^{\,2}} = \frac{3[\gamma\sigma_2^{\,4} + (1-\gamma)\ \sigma_3^{\,4}]}{[\gamma\sigma_2^{\,2} + (1-\gamma)\ \sigma_3^{\,2}]^2} \tag{E.8}$$

Now let H be defined as the ratio of the variance of group number two to the variance of group number one. Thus

$$H = \frac{\sigma_2^{\,2}}{\sigma_3^{\,2}} \tag{E.9}$$

then upon dividing the numerator and denominator by $\sigma_3^{\,4}$, one obtains

$$\beta_2 = \frac{3(1-\gamma+\gamma\ H^2)}{(1-\gamma+\gamma\ H)^2} \tag{E.10}$$

Several limiting cases are immediately obvious, for example

if	$\gamma = 0$ or $\gamma = 1$	then $\beta_2 = 3$
if	$H = 1$	then $\beta_2 = 3$
if	$H \to 0$	then $\beta_2 \to \frac{3}{1-\gamma}$

Figure 30 shows curves of constant β_2; the abscissa is the intermittancy factor γ; the ordinate is the ratio H; the curve for $\beta_2 = 3.5$ is denoted by the symbol □ , the curve for $\beta_2 = 4$ is denoted by the symbol X , the curve for $\beta_2 = 5$ is denoted by the symbol O , the curve for $\beta_2 = 6$ is denoted by the symbol △ , the curve for $\beta_2 = 7$ is denoted by the symbol • .

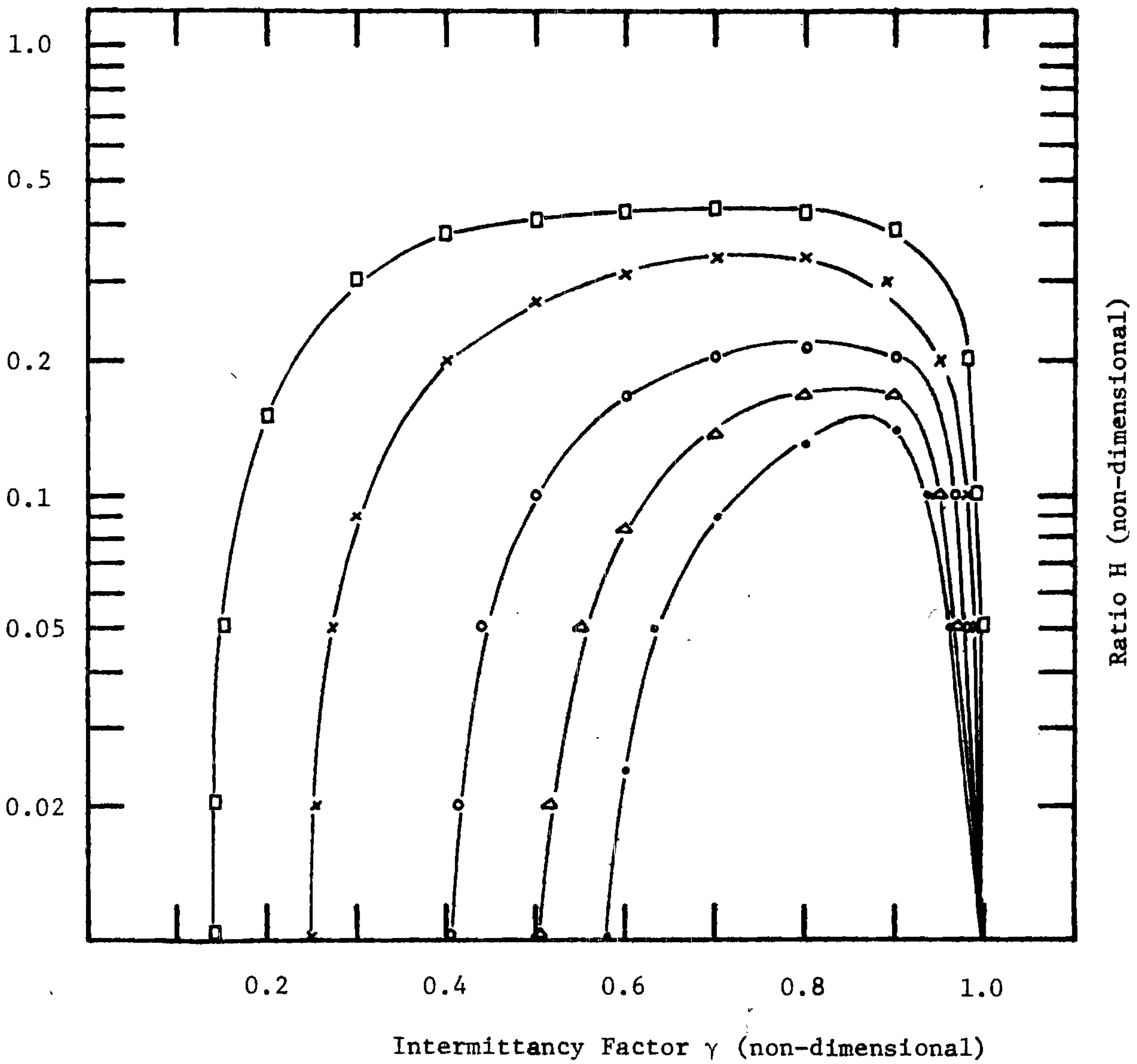

Figure 30. Curves of Constant β_2. The abscissa is the intermittancy factor γ; the ordinate is the ratio H; the curve for $\beta_2 = 3.5$ is denoted by the symbol □, the curve for $\beta_2 = 4$ is denoted by the symbol ✗, the curve for $\beta_2 = 5$ is denoted by the symbol ○, the curve for $\beta_2 = 6$ is denoted by the symbol △, the curve for $\beta_2 = 7$ is denoted by the symbol •.

As can be seen from the figure, if the magnitude of the error is very small in comparison to the magnitude to the turbulent velocities and if simultaneously the number of observations of error alone is approximately equal to or greater than the number of observations of turbulent velocities plus error, then the experimenter could be incorrect in attributing the non-Gaussian character in the form of a seemingly high β_2 to the turbulent velocities alone.

Since the value of the intermittancy factor for the data of the present work is unknown (indeed it is not known if any intermittancy occurs), the observations were not separated into the two types of groups. It has been shown, see Appendix B, that the value of σ_2 is approximately 3 m/sec. From Table 1, one finds that the value of σ_3 is approximately 8.5 m/sec for any of the velocity components. Now consider the worst possible case that could arise in the present work; the case being that every observation of a turbulent component whose magnitude is less than or equal to 3 m/sec be considered as an observation of error alone. For each of the components of the turbulent velocities, this would give rise to a γ of about 0.33. Substitution of the values for σ_2 and σ_3 into equation E.9 gives an $H = 0.125$. Substitution of these values for γ and H into equation E.10 gives $\beta_2 = 4.004$.

Now the values of β_2 given in Table 1 for each of the components was significantly larger (at the 1% level of significance by a t-test) than the β_2 which would arise in the worst case. Therefore, it is very definitely improbable that the non-Gaussian character of the turbulent velocity p.d.f's is due to intermittancy in the set of experimental observations.

APPENDIX F

HERMITE POLYNOMIALS

Univariate Hermite Polynomials

The univariate Hermite polynomials are a well known set of functions which are orthogonal over the interval $(-\infty,\infty)$ with respect to the weight function $\zeta(x)$ where

$$\zeta(x) = e^{-x^2/2} \tag{F.1}$$

A listing of these functions can be found in almost any book dealing with orthogonal polynomials, for example Jackson [1941]. The derivative definition of the Hermite polynomials is given by

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} \zeta(x) \tag{F.2}$$

where n is a non-negative integer.

The relation of recurrence connecting three successive Hermite polynomials is

$$H_{n+1}(x) - xH_n(x) + nH_{n-1}(x) = 0 \tag{F.3}$$

These polynomials satisfy the following orthogonality condition,

$$\int_{-\infty}^{\infty} e^{-x^2/2} H_m(x) H_n(x) \, dx = \begin{cases} 0 & \text{for } n \neq m \\ n! \, (2\pi)^{1/2} & \text{for } n = m \end{cases} \tag{F.4}$$

where n and m are non-negative integers.

The formal expansion of an arbitrary function f(x) in a series of Hermite polynomials is

$$f(x) = \sum_{k=0}^{\infty} C_k H_k(x) \tag{F.5}$$

where

$$C_k = \frac{1}{k! \, (2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-x^2/2} f(x) \, H_k(x) \, dx \tag{F.6}$$

An arbitrary function can also be expanded in a series made of terms of the form $B_k e^{-x^2/2} H_k(x)$. Let F(x) be an arbitrary function, then

$$F(x) = \sum_{k=0}^{\infty} B_k e^{-x^2/2} H_k(x) \tag{F.7}$$

where

$$B_k = \frac{1}{k! (2\pi)^{1/2}} \int_{-\infty}^{\infty} F(x) \, H_k(x) dx \tag{F.8}$$

A series of this form is called a Gram-Charlier series. The Hermite polynomials satisfy the following equations

$$\frac{dH_n(x)}{dx} = n\, H_{n-1}(x) \tag{F.9}$$

$$\frac{d^kH_n(x)}{dx^k} = n(n-1)\ \ldots\ (n-k+1)H_{n-k} \tag{F.10}$$

$$\frac{d^2H_n}{dx^2} - x\,\frac{dH_n}{dx} + nH_n = 0 \tag{F.11}$$

The Hermite polynomials are related to the generating function M(x,t) by the identity

$$M(x,t) = e^{xt - (t^2/2)} = \sum_{k=0}^{\infty} \frac{H_k(x)}{k!}\, t^k \tag{F.12}$$

Finally the first six Hermite polynomials are

$$\begin{aligned} H_0(x) &= 1 \\ H_1(x) &= x \\ H_2(x) &= x^2 - 1 \\ H_3(x) &= x^3 - 3\,x \end{aligned} \tag{F.13}$$

$$H_4(x) = x^4 - 6x^2 + 3$$

$$H_5(x) = x^5 - 10\,x^3 + 15\,x \qquad \text{(F.13 Cont.)}$$

Bivariate Hermite Polynomials

The bivariate Hermite polynomials are not as widely publicized as the univariate ones but several sources of information are available [Kampe´ deFériet [1966] and Appell [1926]].

Consider a positive definite quadratic form

$$\phi(x,y) = ax^2 + 2\,bxy + c\,y^2 \qquad \text{(F.14)}$$

where $a > 0$, and $\Delta = ac - b^2 > 0$

and the form

$$\psi(\xi,\eta) = \frac{c}{\Delta}\,\xi^2 - 2\,\frac{b}{\Delta}\,\xi\eta + \frac{a}{\Delta}\,\eta^2 \qquad \text{(F.15)}$$

where $\xi = ax + by$ and $\eta = bx + cy$ (F.16)

Then $\psi(\xi,\eta) = \phi(x,y)$ (F.17)

The bivariate Hermite polynomials $H_{mn}(x,y)$ and $G_{mn}(x,y)$ are defined by

$$\exp\left\{h(ax + by) + k(bx + cy) - \frac{1}{2}\,\phi(h,k)\right\} = \sum_{m,n=0}^{\infty} \frac{h^m k^n}{m!n!}\,H_{m,n}(x,y) \qquad \text{(F.18)}$$

$$\exp\{hx + ky - \frac{1}{2}\,\psi(h,k)\} = \sum_{m,n=0}^{\infty} \frac{h^m k^n}{m!n!}\, G_{m,n}(x,y) \tag{F.19}$$

where h and k are constants. The derivative definitions of the polynomials are

$$H_{m,n}(x,y) = (-1)^{m+n} \exp\{\frac{1}{2}\,\phi(x,y)\} \frac{\partial^{m+n}}{\partial_x^m\, {}_y^n} [\exp\{-\frac{1}{2}\,\phi(x,y)\}] \tag{F.20}$$

$$G_{m,n}(x,y) = (-1)^{m+n} \exp\{\frac{1}{2}\,\psi(\xi,\eta)\} \frac{\partial^{m+n}}{\partial\xi^m \partial\eta^n} [\exp\{-\frac{1}{2}\,\psi(\xi,\eta)\}] \tag{F.21}$$

The polynomials $H_{m,n}$ and $G_{m,n}$ satisfy the following equations:

$$\frac{\partial}{\partial x} H_{m,n} = amH_{m-1,n} + bnH_{m,n-1} \tag{F.22}$$

$$\frac{\partial}{\partial y} H_{m,n} = bmH_{m-1,n} + cnH_{m,n-1} \tag{F.23}$$

$$\frac{\partial}{\partial x} G_{m,n} = mG_{m-1,n} \tag{F.24}$$

$$\frac{\partial}{\partial y} G_{m,n} = nG_{m,n-1} \tag{F.25}$$

The recursion relations connecting the polynomials are

$$H_{m,n}(x,y) - \xi\, H_{m-1,n}(x,y) + a(m-1)H_{m-2,n}(x,y)$$

$$+ bnH_{m-1,n-1}(x,y) = 0 \tag{F.26}$$

$$H_{m,n}(x,y) - \eta H_{m,n-1}(x,y) + bm\ H_{m-1,n-1}(x,y)$$

$$+ c(n-1)\ H_{m,n-2}(x,y) = 0 \qquad (F.27)$$

$$G_{m,n}(x,y) - x\ G_{m-1,n}(x,y) + \frac{c}{\Delta}\ (m-1)\ G_{m-2,n}(x,y)$$

$$- \frac{b}{\Delta}\ n\ G_{m-1,n-1}(x,y) = 0 \qquad (F.28)$$

$$G_{m,n}(x,y) - y\ G_{m,n-1}(x,y) - \frac{b}{\Delta}\ m\ G_{m-1,n-1}(x,y)$$

$$+ \frac{a}{\Delta}\ (n-1)\ G_{m,n-2}(x,y) = 0 \qquad (F.29)$$

The two sequences of polynomials $H_{m,n}$ and $G_{m,n}$ have the orthogonality condition

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\ \phi(x,y)\right\}\ H_{m,n}(x,y) G_{p,q}(x,y)\, dx\, dy$$

$$= 2\pi\ (\Delta)^{-1/2}\ \ \delta_{m,p}\delta_{n,q} m!n! \qquad (F.30)$$

where δ is the Kronecker delta. Thus the $H_{m,n}$ or the $G_{m,n}$ are not orthogonal to themselves but rather they are biorthogonal, that is they are orthogonal to each other over the (x;y) region $(-\infty,\ \infty;\ -\infty,\ \infty)$ with respect to the weight function $\exp\left\{-\frac{1}{2}\ \phi\right\}$.

An arbitrary function $Q(x,y)$ can be expanded in a series of terms of the form $C_{m,n}\ \exp\left\{-\frac{1}{2}\ \phi\ \right\}\ H_{m,n}$. The expansion is

$$Q(x,y) = \sum_{m,n=0}^{\infty} C_{m,n} \exp\{-\frac{1}{2}\phi\}H_{m,n} \tag{F.31}$$

where

$$C_{m,n} = [2\pi\ (\Delta)^{-1/2} m!n!]^{-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q(x,y)\ G_{mn}(x,y)\,dxdy \tag{F.32}$$

The first few polynomials $H_{m,n}$ are

$$H_{00}(x,y) = 1 \qquad H_{1,0}(x,y) = \xi = ax + by$$

$$H_{0,1}(x,y) = \eta \qquad H_{2,0}(x,y) = \xi^2 - a$$

$$H_{1,1}(x,y) = \xi\eta - b \qquad H_{0,2}(x,y) = \eta^2 - c$$

$$H_{3,0}(x,y) = \xi^3 - 3a\ \xi$$

$$H_{2,1}(x,y) = \xi^2\eta - 2b\xi - a\ \eta \tag{F.33}$$

$$H_{1,2}(x,y) = \xi\eta^2 - 2b\eta - c\xi$$

$$H_{0,3}(x,y) = \eta^3 - 3\ c\eta$$

$$H_{4,0}(x,y) = \xi^4 - 6a\ \xi^2 + 3a^2$$

$$H_{3,1}(x,y) = \xi^3\eta - 3b\xi^2 - 3a\xi\eta + 3ab$$

$$H_{2,2}(x,y) = \xi^2\eta^2 - c\xi^2 - 2b\xi\eta - a\eta^2 + ac + 2b^2$$

$$H_{1,3}(x,y) = \xi\eta^3 - 3b\eta^2 - 3c\xi\eta + 3cb \qquad \text{(F.33 Cont'd)}$$

$$H_{0,4}(x,y) = \eta^4 - 6c\eta^2 + 3c^2$$

where it is assumed that one will substitute for ξ and η in terms of x and y according to equation (F.16). The first few polynomials $G_{m,n}$ are

$$G_{0,0}(x,y) = 1 \qquad G_{1,0}(x,y) = x$$

$$G_{0,1}(x,y) = y \qquad G_{2,0}(x,y) = x^2 - \frac{c}{\Delta}$$

$$G_{1,1}(x,y) = xy + \frac{b}{\Delta} \qquad G_{0,2}(x,y) = y^2 - \frac{a}{\Delta}$$

$$G_{3,0}(x,y) = x^3 - 3\frac{c}{\Delta}x$$

$$G_{2,1}(x,y) = x^2y + 2\frac{b}{\Delta}x - \frac{c}{\Delta}y \qquad \text{(F.34)}$$

$$G_{1,2}(x,y) = xy^2 + 2\frac{b}{\Delta}y - \frac{a}{\Delta}x$$

$$G_{0,3}(x,y) = y^3 - 3\frac{a}{\Delta}y$$

$$G_{4,0}(x,y) = x^4 - 6\frac{c}{\Delta}x^2 + 3\frac{c^2}{\Delta^2}$$

$$G_{3,1}(x,y) = x^3y - \frac{3c}{\Delta}xy + \frac{3b}{\Delta}x^2 - \frac{3bc}{\Delta^2}$$

$$G_{2,2}(x,y) = x^2y^2 - \frac{a}{\Delta}x^2 - \frac{c}{\Delta}y^2 + \frac{4b}{\Delta}xy + \frac{ac}{\Delta^2} + 2\frac{b^2}{\Delta^2}$$

(F.34 Contd)

$$G_{1,3}(x,y) = xy^3 - \frac{3a}{\Delta}xy + \frac{3b}{\Delta}y^2 - \frac{3ac}{\Delta^2}$$

$$G_{0,4}(x,y) = y^4 - 6\frac{a}{\Delta}y^2 + 3\frac{a^2}{\Delta^2}$$

Trivariate Hermite Polynomials

A literature search failed to produce any writing which listed the particular properties of the trivariate Hermite polynomials. However, Appell [1926] has a section which deals with the n-variate case of the Hermite polynomials. By a straightforward but tedious application of the formulas, the particular properties of the trivariate Hermite polynomials were developed.

Definitions

Consider the quadratic form of three variables

$$\phi(v_1,v_2,v_3) = \sum_{i,k=1}^{3} a_{ik}\, v_i v_k \tag{F.35}$$

where

$$a_{ik} = a_{ki} \tag{F.36}$$

and let the quadratic form be positive definite, that is

$$a_{11} > 0, \tag{F.37}$$

and

$$a_{11}a_{22} - a_{12}^2 > 0 \tag{F.37A}$$

and

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} > 0 \tag{F.38}$$

Now the matrix $\underset{\leftrightarrow}{A}$ of equation A.88, which is the inverse of the matrix of variances and covariances, satisfies the positive definite criteria. So let

$$(a_{ik}) = \underset{\leftrightarrow}{A} = \frac{1}{K} \begin{bmatrix} \dfrac{(1-\rho_{23}^2)}{\sigma_1^{\,2}} & -\dfrac{\rho_{12}-\rho_{13}\rho_{23}}{\sigma_1\sigma_2} & -\dfrac{\rho_{13}-\rho_{12}\rho_{23}}{\sigma_1\sigma_3} \\ -\dfrac{\rho_{12}-\rho_{13}\rho_{23}}{\sigma_1\sigma_2} & \dfrac{(1-\rho_{13}^{\,2})}{\sigma_2^2} & -\dfrac{\rho_{23}-\rho_{12}\rho_{13}}{\sigma_2\sigma_3} \\ -\dfrac{\rho_{13}-\rho_{12}\rho_{23}}{\sigma_1\sigma_3} & -\dfrac{\rho_{23}-\rho_{12}\rho_{13}}{\sigma_2\sigma_3} & \dfrac{(1-\rho_{12}^2)}{\sigma_3^{\,2}} \end{bmatrix} \tag{F.39}$$

where $\sigma_1^{\,2}$, $\sigma_2^{\,2}$, and $\sigma_3^{\,2}$ are the variances of v_1, v_2, and v_3; and

ρ_{12}, ρ_{13}, ρ_{23} are the correlation coefficients; and

$$K = 1 - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2 + 2\,\rho_{12}\rho_{13}\rho_{23} \tag{F.40}$$

Now
$$|A| = (K\sigma_1^2\sigma_2^2\sigma_3^2)^{-1} = \Delta \tag{F.41}$$

The trivariate Hermite polynomials are defined by

$$\exp\{-\tfrac{1}{2}\,\phi(v_1-h_1,\ v_2-h_2,\ v_3-h_3)\} =$$

$$\exp\{-\tfrac{1}{2}\phi(v_1,v_2,v_3)\} \sum_{m_1,m_2,m_3=0}^{\infty} \frac{h_1^{m_1}}{m_1!}\ \frac{h_2^{m_2}}{m_2!}\ \frac{h_3^{m_3}}{m_3!}\ H_{m_1,m_2,m_3}(v_1,v_2,v_3) \tag{F.42}$$

(where h_1,h_2 and h_3 are arbitrary constants), or by the equivalent formula

$$\exp\{\tfrac{1}{2}[h_1 \frac{\partial\phi(v_1,v_2,v_3)}{\partial v_1} + h_2 \frac{\partial\,\phi(v_1,v_2,v_3)}{\partial v_2} + h_3 \frac{\partial\,\phi(v_1,v_2,v_3)}{\partial v_3}]$$

$$- \tfrac{1}{2}\,\phi(h_1,h_2,h_3)\} = \sum_{m_1,m_2,m_3=0}^{\infty} \frac{h_1^{m_1}}{m_1!}\ \frac{h_2^{m_2}}{m_2!}\ \frac{h_3^{m_3}}{m_3!}\ H_{m_1,m_2,m_3}(v_1,v_2,v_3) \tag{F.43}$$

Now one introduces the new parameters k_1, k_2, and k_3 which are defined as functions of h_1, h_2, h_3 such that

$$k_1 = \frac{1}{2} \frac{\partial \phi(h_1,h_2,h_3)}{\partial h_1} \tag{F.44}$$

$$k_2 = \frac{1}{2} \frac{\partial \phi(h_1,h_2,h_3)}{\partial h_2} \tag{F.45}$$

$$k_3 = \frac{1}{2} \frac{\partial \phi(h_1,h_2,h_3)}{\partial h_3} \tag{F.46}$$

Consider the cofactor matrix of the matrix A,

$$(b_{ik}) = \frac{1}{K} \begin{bmatrix} \dfrac{1}{\sigma_2^2\sigma_3^2} & \dfrac{\rho_{12}}{\sigma_1\sigma_2\sigma_3^2} & \dfrac{\rho_{13}}{\sigma_1\sigma_2^2\sigma_3} \\ \dfrac{\rho_{12}}{\sigma_1\sigma_2\sigma_3^2} & \dfrac{1}{\sigma_1^2\sigma_3^2} & \dfrac{\rho_{23}}{\sigma_1^2\sigma_2\sigma_3} \\ \dfrac{\rho_{13}}{\sigma_1\sigma_2^2\sigma_3} & \dfrac{\rho_{23}}{\sigma_1^2\sigma_2\sigma_3} & \dfrac{1}{\sigma_1^2\sigma_2^2} \end{bmatrix} \tag{F.47}$$

where all of the terms have previously been defined, and denote (b_{ik}) by B. Since A is symmetric, B is also the adjoint of A. Next one introduces the function $\psi(\xi_1,\xi_2,\xi_3)$ which is defined by (ψ is the quadratic adjoint of ϕ)

$$\psi(\xi_1,\xi_2,\xi_3) = \sum_{i,k=0}^{3} \frac{b_{ik}}{\Delta} \xi_i \xi_k \tag{F.48}$$

$$= \sigma_1^2 \xi_1^2 + 2\rho_{12}\sigma_1\sigma_2\xi_1\xi_2$$

$$+ 2 \rho_{13}\sigma_1\sigma_3\xi_1\xi_3 + \sigma_2^2\xi_2^2$$

$$+ 2 \rho_{23}\sigma_2\sigma_3\xi_2\xi_3 + \sigma_3^2\xi_3^2 \tag{F.49}$$

where ξ_1, ξ_2, and ξ_3 are defined to be functions of v_1, v_2, and v_3 such that

$$\xi_1 = \frac{1}{2} \frac{\partial \phi(v_1,v_2,v_3)}{\partial v_1} \tag{F.50}$$

$$\xi_2 = \frac{1}{2} \frac{\partial \phi(v_1,v_2,v_3)}{\partial v_2} \tag{F.51}$$

$$\xi_3 = \frac{1}{2} \frac{\partial \phi(v_1,v_2,v_3)}{\partial v_3} \tag{F.52}$$

or

$$\xi_1 = \frac{1}{K} [\frac{(1-\rho_{23}^2)}{\sigma_1^2} v_1 - \frac{(\rho_{12}-\rho_{13}\rho_{23})}{\sigma_1\sigma_2} v_2$$

$$- \frac{(\rho_{13}-\rho_{12}\rho_{23})}{\sigma_1\sigma_3} v_3] \qquad \text{(F.53)}$$

$$\xi_2 = \frac{1}{K}[\frac{(\rho_{12}-\rho_{13}\rho_{23})}{\sigma_1\sigma_2} v_1 + \frac{(1-\rho_{13}^2)}{\sigma_2^2} v_2 - \frac{(\rho_{23}-\rho_{12}\rho_{13})}{\sigma_2\sigma_3} v_3] \qquad \text{(F.54)}$$

$$\xi_3 = \frac{1}{K} [- \frac{(\rho_{13}-\rho_{12}\rho_{23})}{\sigma_1\sigma_3} v_1 - \frac{(\rho_{23}-\rho_{12}\rho_{13})}{\sigma_2\sigma_3} v_2 + \frac{(1-\rho_{12}^2)}{\sigma_3^2} v_3] \qquad \text{(F.55)}$$

Thus in the sense that the ξ's are functions of v's, the function ψ can be considered as $\psi(v_1,v_2,v_3)$. It should be noted that the constants h_1,h_2,h_3 can be expressed as

$$h_1 = \frac{1}{2} \frac{\partial\psi(k_1,k_2,k_3)}{\partial k_1} \qquad \text{(F.56)}$$

$$h_2 = \frac{1}{2} \frac{\partial\psi(k_1,k_2,k_3)}{\partial k_2} \qquad \text{(F.57)}$$

$$h_3 = \frac{1}{2} \frac{\partial\psi(k_1,k_2,k_3)}{\partial k_3} \qquad \text{(F.58)}$$

and the variables v_1,v_2,v_3 can be expressed as

$$v_1 = \frac{1}{2} \frac{\partial\psi(\xi_1,\xi_2,\xi_3)}{\partial \xi_1} \qquad \text{(F.59)}$$

$$v_2 = \frac{1}{2} \frac{\partial\psi\ (\xi_1, \xi_2, \xi_3)}{\partial\ \xi_2} \tag{F.60}$$

$$v_3 = \frac{1}{2} \frac{\partial\ \psi(\xi_1, \xi_2, \xi_3)}{\partial\ \xi_3} \tag{F.61}$$

or

$$v_1 = \sigma_1{}^2\xi_1 + \rho_{12}\sigma_1\sigma_2\xi_2 + \rho_{13}\sigma_1\sigma_3\xi_3 \tag{F.62}$$

$$v_2 = \rho_{12}\sigma_1\sigma_2\xi_1 + \sigma_2{}^2\xi_2 + \rho_{23}\sigma_2\sigma_3\xi_3 \tag{F.63}$$

$$v_3 = \rho_{13}\sigma_1\sigma_3\xi_1 + \rho_{23}\sigma_2\sigma_3\xi_2 + \sigma_3^2\ \xi_3 \tag{F.64}$$

Also since ϕ and ψ are adjoints

$$\phi(h_1, h_2, h_3) = \psi(k_1, k_2, k_3) \tag{F.65}$$

and
$$\phi(v_1, v_2, v_3) = \psi(v_1, v_2, v_3) \tag{F.66}$$

where it is understood that one is to replace the ξ's in equation F.49 with their functions of v's given by F.50, F.51, and F.52. The complementary set of polynomials $G_{m_1,m_2,m_3}(v_1, v_2, v_3)$ are defined by

$$\exp\{-\tfrac{1}{2}\ \psi\ (\xi_1-k_1, \xi_2-k_2, \xi_3-k_3)\}$$

$$= \exp\{-\frac{1}{2}\psi(\xi_1,\xi_2,\xi_3)\} \sum_{m_1,m_2,m_3=0}^{\infty} \frac{k_1^{m_1}}{m_1!}\frac{k_2^{m_2}}{m_2!}\frac{k_3^{m_3}}{m_3!} G_{m_1,m_2,m_3}(v_1,v_2v_3) \quad (F.67)$$

where it is understood that one is to replace the ξ_i's with the proper function of the v_i's. An equivalent form for F.67 is

$$\exp\{\frac{1}{2}[k_1 \frac{\partial\,\psi(\xi_1,\xi_2,\xi_3)}{\partial\,\xi_1} + k_2 \frac{\partial\,\psi(\xi_1,\xi_2,\xi_3)}{\partial\,\xi_2} + k_3 \frac{\partial\,\psi(\xi_1,\xi_2,\xi_3)}{\partial\,\xi_3}]$$

$$-\frac{1}{2}\psi(k_1,k_2,k_3)\} = \sum_{m_1,m_2,m_3=0}^{\infty} \frac{k_1^{m_1}}{m_1!}\frac{k_2^{m_2}}{m_2!}\frac{k_3^{m_3}}{m_3!} G_{m_1,m_2,m_3}(v_1,v_2,v_3) \quad (F.68)$$

It should be noted that only in the case that

$$\phi(v_1,v_2,v_3) = v_1^2 + v_2^2 + v_3^2 = \psi(v_1,v_2,v_3) \quad (F.69)$$

does one have the factorization into univariate Hermite polynomials, that is

$$H_{m_1,m_2,m_3}(v_1,v_2,v_3) = G_{m_1,m_2,m_3}(v_1,v_2,v_3) = H_{m_1}(v_1)\,H_{m_2}(v_2)\,H_{m_3}(v_3) \quad (F.70)$$

The above case amounts to the correlation coefficients being zero, and the variables v_1, v_2, and v_3 being statistically independent.

Now the derivative definitions of $H_{m_1,m_2,m_3}(v_1,v_2,v_3)$ and $G_{m_1,m_2,m_3}(v_1,v_2,v_3)$ are

$$H_{m_1,m_2,m_3}(v_1,v_2,v_3) = [(-1)^{m_1+m_2+m_3} \exp\{\tfrac{1}{2}\,\phi(v_1,v_2,v_3)\}]$$

$$\cdot [\frac{\partial^{m_1+m_2+m_3}}{\partial v_1^{m_1}\,\partial v_2^{m_2}\,\partial v_3^{m_3}} (\exp\{-\tfrac{1}{2}\,\phi(v_1,v_2,v_3)\})] \qquad (F.71)$$

and

$$G_{m_1,m_2,m_3}(v_1,v_2,v_3) = [(-1)^{m_1+m_2+m_3} \exp\{\tfrac{1}{2}\,\psi(\xi_1,\xi_2,\xi_3)\}]$$

$$\cdot [\frac{\partial^{m_1+m_2+m_3}}{\partial \xi_1^{m_1}\,\partial \xi_2^{m_2}\,\partial \xi_3^{m_3}} (\exp\{-\tfrac{1}{2}\,\psi(\xi_1,\xi_2,\xi_3)\})] \qquad (F.72)$$

It is obvious that $H_{0,0,0}(v_1,v_2,v_3) = G_{0,0,0}(v_1,v_2,v_3) = 1$

It can be shown that (see Appell [1926] pages 367-369) $H_{m_1,m_2,m_3}(v_1,v_2,v_3)$ and $G_{m_1,m_2,m_3}(v_1,v_2,v_3)$ obey the orthogonality relation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\exp\{-\tfrac{1}{2}\,\phi(v_1,v_2,v_3)\}\, H_{m_1,m_2,m_3}(v_1,v_2,v_3)]$$

$$\cdot[G_{q_1,q_2,q_3}(v_1,v_2,v_3)]\; dv_1 dv_2 dv_3$$

$$= (2\pi)^{3/2}\,(\Delta)^{-1/2}\,\delta_{m_1,q_1}\delta_{m_2,q_2}\delta_{m_3,q_3}\, m_1!\, m_2!\, m_3! \tag{F.73}$$

that is H_{m_1,m_2,m_3} and G_{q_1,q_2,q_3} are biorthogonal over the $(v_1;v_2;v_3)$ region $(-\infty,\infty;-\infty;-\infty,\infty)$ with respect to the weight function $\exp\{-\frac{1}{2}\,\phi(v_1,v_2,v_3)\}$.

Integral Expressions for H_{m_1,m_2,m_3} and G_{m_1,m_2,m_3}

Consider the case where $m_1 = m_2 = m_3 = q_1 = q_2 = q_3 = 0$, then equation F.73 becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}\,\phi(v_1,v_2,v_3)\}dv_1 dv_2 dv_3 = (2\pi)^{3/2}\,(\Delta)^{-1/2} \tag{F.74}$$

making the change of variables to

$$u\text{-}ih_1, \qquad \nu\text{-}ih_2, \qquad w\text{-}ih_3 \tag{F.75}$$

where i is the pure imaginary quantity and the h_1,h_2,h_3 have previously been defined, then

$$dv_1 = du, \qquad dv_2 = d\nu \qquad , \text{ and } dv_3 = dw$$

Now
$$-\frac{1}{2}\,\phi\,(u - ih_1,\ \nu - ih_2,\ w - ih_3) =$$

$$-\frac{1}{2}\,\{a_{11}(u-ih_1)^2 + 2a_{12}(u-ih_1)(\nu-ih_2) + a_{22}(\nu-ih_2)^2$$

$$+ 2a_{13}(u-ih_1)(w-ih_3) + 2a_{23}(\nu-ih_2)(w-ih_3)$$

$$+ a_{33}(w-ih_3)^2\}$$

which after some rearrangement becomes

$$= -\frac{1}{2}\,\phi(u,\nu,w) + \frac{1}{2}\,\phi(h_1,h_2,h_3) + i[u(a_{11}h_1 + a_{12}h_2 + a_{13}h_3)$$

$$+ \nu(a_{12}h_1 + a_{22}h_2 + a_{23}h_3) + w(a_{13}h_1 + a_{23}h_2 + a_{33}h_3)]$$

but from equations F.44, F.45, F.46, and F.65 one has

$$= -\frac{1}{2}\,\phi(u,\nu,w) + \frac{1}{2}\,\psi(k_1,k_2,k_3) + i(k_1u + k_2\nu + k_3w)$$

substitution of the above into equation F.74 after the change of variable F.75 gives

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}\,\phi(u,\nu,w) + \tfrac{1}{2}\,\psi(k_1,k_2,k_3)$$

$$+\, i(k_1 u + k_2 \nu + k_3 w)\}\,du\,d\nu\,dw = (2\pi)^{3/2}(\Delta)^{-1/2} \qquad \text{(F.76)}$$

Now multiply both sides of F.76 by

$$(2\pi)^{-3/2}\,(\Delta)^{1/2}\,\exp\{k_1 v_1 + k_2 v_2 + k_3 v_3 - \tfrac{1}{2}\,\psi(k_1,k_2,k_3)\}$$

to obtain (and recall F.59, F.60, and F.61)

$$\exp\{\tfrac{1}{2}[k_1 \frac{\partial\,\psi(\xi_1,\xi_2,\xi_3)}{\partial\,\xi_1} + k_2 \frac{\partial\,\psi(\xi_1,\xi_2,\xi_3)}{\partial\,\xi_2} + k_3 \frac{\partial\,\psi(\xi_1,\xi_2,\xi_3)}{\partial\,\xi_3}]$$

$$-\tfrac{1}{2}\,\psi(k_1,k_2,k_3)\} = (2\pi)^{-3/2}\,(\Delta)^{1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}\,\phi(u,\nu,w)$$

$$+\, k_1(v_1 + iu) + k_2(v_2 + i\nu) + k_3(v_3 + iw)\}\,du\,d\nu\,dw \qquad \text{(F.77)}$$

Now the left hand side of equation F.77 is the left hand side of equation F.68 and recall that

$$\exp\{k_1(v_1 + iu)\} = \sum_{m_1=0}^{\infty} \frac{k_1^{\,m_1}(v_1 + iu)^{m_1}}{m_1!} \qquad \text{(F.78)}$$

$$\exp\, k_2(v_2 + i\nu) = \sum_{m_2=0}^{\infty} \frac{k_2^{\,m_2}(v_2 + i\nu)^{m_2}}{m_2!} \qquad \text{(F.79)}$$

and

$$\exp\{k_3(v_3 + iw)\} = \sum_{m_3=0}^{\infty} \frac{k_3^{m_3}(v_3 + iw)^{m_3}}{m_3!} \qquad \text{(F.80)}$$

So substitution from equations F.68, F.78, F.79, and F.80 into equation F.77 gives

$$\sum_{m_1,m_2,m_3=0}^{\infty} \frac{k_1^{m_1}}{m_1!} \frac{k_2^{m_2}}{m_2!} \frac{k_3^{m_3}}{m_3!} \; G_{m_1,m_2,m_3}(v_1,v_2,v_3)$$

$$= \sum_{m_1,m_2,m_3=0}^{\infty} \frac{k_1^{m_1}}{m_1!} \frac{k_2^{m_2}}{m_2!} \frac{k_3^{m_3}}{m_3!} \; (2\pi)^{-3/2} \, (\Delta)^{1/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}$$

$$(v_1 + iu)^{m_1} \, (v_2 + i\nu)^{m_2} \, (v_3 + iw)^{m_3} \exp\{- \tfrac{1}{2} \, \phi(u,\nu,w)\} du d\nu dw \qquad \text{(F.81)}$$

and by equating terms of like powers in k_1,k_2, and k_3 one obtains the integral expression for $G_{m_1,m_2,m_3}(v_1,v_2,v_3)$, that is

$$G_{m_1,m_2,m_3}(v_1,v_2,v_3) = (2\pi)^{-3/2} \, (\Delta)^{1/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (v_1$$

$$+ iu)^{m_1}(v_2 + i\nu)^{m_2}(v_3 + iw)^{m_3} \exp\{- \tfrac{1}{2} \, \phi(u,\nu,w)\} \, du d\nu dw \qquad \text{(F.82)}$$

By an exactly analogous argument one finds

$$H_{m_1,m_2,m_3}(v_1,v_2,v_3) = (2\pi)^{-3/2} (\Delta)^{1/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\xi_1$$

$$+ iu)^{m_1} (\xi_2 + i\nu)^{m_2} (\xi_3 + iw)^{m_3} \exp\{-\frac{1}{2}\psi(u,\nu,w)\} du d\nu dw \qquad \text{(F.83)}$$

Now if one differentiates equation F.82 with respect to v_1 using Leibnitz's rule, one has

$$\frac{G_{m_1,m_2,m_3}(v_1,v_2,v_3)}{\partial v_1} = m_1 G_{m_1-1,m_2,m_3} \qquad \text{(F.84)}$$

and similarly

$$\frac{\partial G_{m_1,m_2,m_3}(v_1,v_2,v_3)}{\partial v_2} = m_2 G_{m_1,m_2-1,m_3} \qquad \text{(F.85)}$$

$$\frac{\partial G_{m_1,m_2,m_3}(v_1,v_2,v_3)}{\partial v_3} = m_3 G_{m_1,m_2,m_3-1} \qquad \text{(F.86)}$$

Likewise differentiating equation F.83 and recalling equation F.53, F.54, and F.55, one has

$$\frac{\partial H_{m_1,m_2,m_3}(v_1,v_2,v_3)}{\partial v_1} = m_1 a_{11} H_{m_1-1,m_2,m_3}$$

$$+ m_2 a_{12} H_{m_1,m_2-1,m_3} + m_3 a_{13} H_{m_1,m_2,m_3-1} \qquad (F.87)$$

where a_{ij} represents elements of matrix A given by equation A.88. Similarly, from differentiation with respect to v_2 and v_3 one has

$$\frac{\partial H_{m_1,m_2,m_3}(v_1,v_2,v_3)}{\partial v_2} = m_1 a_{12} H_{m_1-1,m_2,m_3}$$

$$+ m_2 a_{22} H_{m_1,m_2-1,m_3} + m_3 a_{23} H_{m_1,m_2,m_3-1} \qquad (F.88)$$

$$\frac{\partial H_{m_1,m_2,m_3}(v_1,v_2,v_3)}{\partial v_3} = m_1 a_{13} H_{m_1-1,m_2,m_3}$$

$$+ m_2 a_{23} H_{m_1,m_2-1,m_3} + m_3 a_{33} H_{m_1,m_2,m_3-1} \qquad (F.89)$$

Recursion Relations

Substitution into equation F.43 from equations F.50, F.51, and F.52 gives

$$\exp\{h_1\xi_1 + h_2\xi_2 + h_3\xi_3 - \frac{1}{2}\phi(h_1,h_2,h_3)\}$$

$$= \sum_{m_1,m_2,m_3=0}^{\infty} \frac{h_1^{m_1}}{m_1!} \frac{h_2^{m_2}}{m_2!} \frac{h_3^{m_3}}{m_3!} H_{m_1,m_2,m_3}(v_1,v_2,v_3) \tag{F.90}$$

Taking derivatives of both sides with respect to h_1 gives

$$(\xi_1-a_{11}h_1-a_{12}h_2-a_{13}h_3) \sum_{m_1,m_2,m_3=0} \frac{h_1^{m_1}}{m_1!} \frac{h_2^{m_2}}{m_2!} \frac{h_3^{m_3}}{m_3!} H_{m_1,m_2,m_3}(v_1,v_2,v_3)$$

$$= \sum_{\substack{m_1=1\\ m_2,m_3=0}}^{\infty} \frac{h_1^{(m_1-1)}}{(m_1-1)!} \frac{h_2^{m_2}}{m_2!} \frac{h_3^{m_3}}{m_3!} H_{m_1,m_2,m_3}(v_1,v_2,v_3) \tag{F.91}$$

Equating coefficients of $h_1^{m-1}\ h_2^{m_2}\ h_3^{m_3}$ one obtains

$$\frac{\xi_1 H_{m_1-1,m_2,m_3}}{(m_1-1)!m_2!m_3!} - \frac{a_{11}H_{m_1-2,m_2,m_3}}{(m_1-2)!m_2!m_3!} - \frac{a_{12}H_{m_1-1,m_2-1,m_3}}{(m_1-1)!(m_2-1)!m_3!}$$

$$- \frac{a_{13}H_{m_1-1,m_2,m_3-1}}{(m_1-1)!m_2!(m_3-1)!} = \frac{H_{m_1,m_2,m_3}}{(m_1-1)!m_2m_3!} \tag{F.92}$$

Multiplying both sides by $(m_1-1)!m_2!m_3!$ and rearranging gives

$$H_{m_1,m_2,m_3} - \xi_1 H_{m_1-1,m_3,m_3} + a_{11}(m_1-1) H_{m_1-2,m_2,m_3}$$

$$+ a_{12} m_2 H_{m_1-1,m_2-1,m_3} + a_{13} m_3 H_{m_1-1,m_2,m_3-1} = 0 \tag{F.93}$$

In an analogous manner one obtains

$$H_{m_1,m_2,m_3} - \xi_2 H_{m_1,m_2-1,m_3} + a_{12} m_1 H_{m_1-1,m_2-1,m_3}$$

$$+ a_{22}(m_2-1) H_{m_1,m_2-2,m_3} + a_{23} m_3 H_{m_1,m_2-1,m_3-1} = 0 \tag{F.94}$$

$$H_{m_1,m_2,m_3} - \xi_3 H_{m_1,m_2,m_3-1} + a_{13} m_1 H_{m_1-1,m_2,m_3-1}$$

$$+ a_{23} m_2 H_{m_1,m_2-1,m_3-1} + a_{33}(m_3-1) H_{m_1,m_2,m_3-2} = 0 \tag{F.95}$$

By a similar argument, using F.68 and taking derivatives with respect to k_1, k_2, and k_3 one obtains

$$G_{m_1,m_2,m_3} - v_1 G_{m_1-1,m_2,m_3} + \frac{b_{11}}{\Delta}(m_1-1) G_{m_1-2,m_2,m_3}$$

$$+ \frac{b_{12}}{\Delta} m_2 G_{m_1-1,m_2-1,m_3} + \frac{b_{13}}{\Delta} m_3 G_{m_1-1,m_2,m_3-1} = 0 \tag{F.96}$$

$$G_{m_1,m_2,m_3} - v_2 G_{m_1,m_2-1,m_3} + \frac{b_{12}}{\Delta} m_1 G_{m_1-1,m_2-1,m_3}$$

$$+ \frac{b_{22}}{\Delta} (m_2-1) G_{m_1,m_2-2,m_3} + \frac{b_{23}}{\Delta} m_3 G_{m_1,m_2-1,m_3-1} = 0 \qquad \text{(F.97)}$$

$$G_{m_1,m_2,m_3} - v_3 G_{m_1,m_2,m_3-1} + \frac{b_{13}}{\Delta} m_1 G_{m_1-1,m_2,m_3-1}$$

$$+ \frac{b_{23}}{\Delta} m_2 G_{m_1,m_2-1,m_3-1} + \frac{b_{33}}{\Delta} (m_3-1) G_{m_1,m_2,m_3-2} = 0 \qquad \text{(F.98)}$$

Formulas for the First Few Polynomials

Using the definitions or the recursion relations, one arrives at the following formulas:

$$H_{0,0,0} = 1$$

$$H_{1,0,0} = \frac{1}{K\sigma_1} [(1-\rho_{23}^2) \frac{v_1}{\sigma_1} - (\rho_{12}-\rho_{13}\rho_{23}) \frac{v_2}{\sigma_2} - (\rho_{13}-\rho_{12}\rho_{23}) \frac{v_3}{\sigma_3}] = \xi_1$$

$$\text{(F.99)}$$

$$H_{0,1,0} = \frac{1}{K\sigma_2} [(1-\rho_{13}^2) \frac{v_2}{\sigma_2} - (\rho_{12}-\rho_{13}\rho_{23}) \frac{v_1}{\sigma_1} - (\rho_{23}-\rho_{12}\rho_{13}) \frac{v_3}{\sigma_3}] = \xi_2$$

$$H_{0,0,1} = \frac{1}{K\sigma_3} [(1-\rho_{12}^2) \frac{v_3}{\sigma_3} - (\rho_{13}-\rho_{12}\rho_{23}) \frac{v_1}{\sigma_1} - (\rho_{23}-\rho_{12}\rho_{13}) \frac{v_2}{\sigma_2}] = \xi_3$$

$$H_{1,1,0} = \frac{(\rho_{12}-\rho_{13}\rho_{23})}{K\sigma_1\sigma_2} + \frac{1}{K^2\sigma_1\sigma_2}\left[(1-\rho_{13}^2)\,\frac{v_2}{\sigma_2} - (\rho_{12}-\rho_{13}\rho_{23})\,\frac{v_1}{\sigma_1}\right.$$

$$\left. -(\rho_{23}-\rho_{12}\rho_{13})\,\frac{v_3}{\sigma_3}\right]\left[(1-\rho_{23}^2)\,\frac{v_1}{\sigma_1} - (\rho_{12}-\rho_{13}\rho_{23})\frac{v_2}{\sigma_2} - (\rho_{13}-\rho_{12}\rho_{23})\frac{v_3}{\sigma_3}\right]$$

$$H_{1,1,0} = \frac{(\rho_{12}-\rho_{13}\rho_{23})}{K\sigma_1\sigma_2} + \xi_1\xi_2$$

$$H_{1,0,1} = \frac{(\rho_{13}-\rho_{12}\rho_{23})}{K\sigma_1\sigma_3} + \xi_1\xi_3$$

$$H_{0,1,1} = \frac{(\rho_{23}-\rho_{12}\rho_{13})}{K\sigma_2\sigma_3} + \xi_2\xi_3 \qquad \text{(F.99 Cont.)}$$

$$H_{2,0,0} = \xi_1^2 - \frac{(1-\rho_{23}^2)}{K\,\sigma_1^{\,2}}$$

$$H_{0,2,0} = \xi_2^2 - \frac{(1-\rho_{13}^2)}{K\,\sigma_2^{\,2}}$$

$$H_{0,0,2} = \xi_3^2 - \frac{(1-\rho_{12}^2)}{K\,\sigma_3^{\,2}}$$

$$H_{3,0,0} = \xi_1^3 - \frac{3(1-\rho_{23}^2)}{K\,\sigma_1^{\,2}}\;\xi_1$$

$$H_{0,3,0} = \xi_2^3 - \frac{3(1-\rho_{13}^2)}{K\sigma_2^2}\ \xi_2$$

$$H_{0,0,3} = \xi_3^3 - \frac{3(1-\rho_{12}^2)}{K\sigma_3^2}\ \xi_3$$

$$H_{2,1,0} = \xi_1^2\xi_2 + \frac{2(\rho_{12}-\rho_{13}\rho_{23})}{K\sigma_1\sigma_2}\ \xi_1 - \frac{(1-\rho_{23}^2)}{K\sigma_1^2}\ \xi_2$$

$$H_{1,2,0} = \xi_1\xi_2^2 + \frac{2(\rho_{12}-\rho_{13}\rho_{23})}{K\sigma_1\sigma_2}\ \xi_2 - \frac{(1-\rho_{13}^2)}{K\ \sigma_2^2}\ \xi_1$$

$$H_{2,0,1} = \xi_1^2\xi_3 + \frac{2(\rho_{13}-\rho_{12}\rho_{23})}{K\sigma_1\sigma_3}\ \xi_1 - \frac{(1-\rho_{23}^2)}{K\sigma_1^2}\ \xi_3 \qquad \text{(F.99 Cont.)}$$

$$H_{1,0,2} = \xi_1\xi_3^2 + \frac{2(\rho_{13}-\rho_{12}\rho_{23})}{K\sigma_1\sigma_3}\ \xi_3 - \frac{(1-\rho_{12}^2)}{K\sigma_3^2}\ \xi_1$$

$$H_{0,2,1} = \xi_2^2\xi_3 + \frac{2(\rho_{23}-\rho_{12}\rho_{13})}{K\sigma_2\sigma_3}\ \xi_2 - \frac{(1-\rho_{13}^2)}{K\ \sigma_2^2}\ \xi_3$$

$$H_{0,1,2} = \xi_2\xi_3^2 + \frac{2(\rho_{23}-\rho_{12}\rho_{13})}{K\sigma_2\sigma_3}\ \xi_3 - \frac{(1-\rho_{12}^2)}{K\ \sigma_3^2}\ \xi_2$$

$$H_{1,1,1} = \xi_1\xi_2\xi_3 + \frac{(\rho_{23}-\rho_{12}\rho_{13})}{K\ \sigma_2\sigma_3}\ \xi_1 + \frac{(\rho_{13}-\rho_{12}\rho_{23})}{K\sigma_1\sigma_3}\ \xi_2$$

$$+ \frac{(\rho_{12}-\rho_{13}\rho_{23})}{K\ \sigma_1\sigma_2} \quad \xi_3$$

$$H_{4,0,0} = \xi_1^4 - 6\,\frac{(1-\rho_{23}^2)}{K\ \sigma_1^{\ 2}}\quad \xi_1^2 \ + \ 3\left[\frac{(1-\rho_{23}^2)}{K\sigma_1^{\ 2}}\right]^2$$

$$H_{0,4,0} = \xi_2^4 - 6\,\frac{(1-\rho_{13}^2)}{K\sigma_2^{\ 2}}\quad \xi_2^2 \ + \ 3\left[\frac{(1-\rho_{13}^2)}{K\ \sigma_2^{\ 2}}\right]^2$$

$$H_{0,0,4} = \xi_3^4 - 6\,\frac{(1-\rho_{12}^2)}{K\sigma_3^{\ 2}}\quad \xi_3^2 \ + \ 3\left[\frac{(1-\rho_{12}^2)}{K\ \sigma_3^{\ 2}}\right]^2$$

$$H_{3,1,0} = \xi_1^3\xi_2 + 3\,\frac{(\rho_{12}-\rho_{13}\rho_{23})}{K\ \ \sigma_1\sigma_2}\,\xi_1^2 \ - 3\,\frac{(1-\rho_{23}^2)}{K\ \sigma_1^{\ 2}}\,\xi_1\xi_2 \qquad \text{(F.99 Cont.)}$$

$$- \ 3\left[\frac{1-\rho_{23}^2}{K\sigma_1^{\ 2}}\right]\left[\frac{\rho_{12}-\rho_{13}\rho_{23}}{K\ \sigma_1\sigma_2}\right]$$

$$H_{3,0,1} = \xi_1^3\xi_3 + 3\,\frac{(\rho_{13}-\rho_{12}\rho_{23})}{K\ \sigma_1\sigma_3}\quad \xi_1^2 \ - \ 3\,\frac{(1-\rho_{23}^2)}{K\sigma_1^{\ 2}}\,\xi_1\xi_3$$

$$- \ 3\left[\frac{1-\rho_{23}^2}{K\sigma_1^{\ 2}}\right]\left[\frac{\rho_{13}-\rho_{12}\rho_{23}}{K\ \sigma_1\sigma_3}\right]$$

$$H_{1,3,0} = \xi_2^3\xi_1 + 3\,\frac{(\rho_{12}-\rho_{13}\rho_{23})}{K\ \sigma_1\sigma_2}\quad \xi_2^2 \ - \ 3\,\frac{(1-\rho_{13}^2)}{K\sigma_2^{\ 2}}\,\xi_1\xi_2$$

$$- 3\left[\frac{1-\rho_{13}^2}{K\sigma_2^{\,2}}\right]\left[\frac{\rho_{12}-\rho_{13}\rho_{23}}{K\ \sigma_1\sigma_2}\right]$$

$$H_{1,0,3} = \xi_3^3\xi_1 + 3\,\frac{(\rho_{13}-\rho_{12}\rho_{23})}{K\ \sigma_1\sigma_3}\ \xi_3^2 - 3\,\frac{(1-\rho_{12}^2)}{K\ \sigma_3^{\,2}}\ \xi_1\xi_3$$

$$- 3\left[\frac{1-\rho_{12}^2}{K\sigma_3^2}\right]\left[\frac{\rho_{13}-\rho_{12}\rho_{23}}{K\ \sigma_1\sigma_3}\right]$$

$$H_{0,1,3} = \xi_3^3\xi_2 + 3\,\frac{(\rho_{23}-\rho_{12}\rho_{13})}{K\ \sigma_2\sigma_3}\ \xi_3^2 - 3\ \frac{(1-\rho_{12}^2)}{K\sigma_3^2}\ \xi_2\xi_3$$

$$- 3\left[\frac{1-\rho_{12}^2}{K\sigma_3^2}\right]\left[\frac{\rho_{23}-\rho_{12}\rho_{13}}{K\ \sigma_2\sigma_3}\right] \qquad \text{(F.99 Cont.)}$$

$$H_{0,3,1} = \xi_2^3\xi_3 + 3\ \frac{(\rho_{23}-\rho_{12}\rho_{13})}{K\ \sigma_2\sigma_3}\ \xi_2^2 - 3\,\frac{(1-\rho_{13}^2)}{K\sigma_2^{\,2}}\ \xi_2\xi_3$$

$$- 3\left[\frac{1-\rho_{13}^2}{K\sigma_2^2}\right]\left[\frac{\rho_{23}-\rho_{12}\rho_{13}}{K\ \sigma_2\sigma_3}\right]$$

$$H_{2,2,0} = \xi_1^2\xi_2^2 - \frac{(1-\rho_{13}^2)}{K\sigma_2^{\,2}}\ \xi_1^2 - \frac{(1-\rho_{23}^2)}{K\sigma_1^{\,2}}\ \xi_2^2$$

$$+ 4\,\frac{(\rho_{12}-\rho_{13}\rho_{23})}{K\ \sigma_1\sigma_2}\ \xi_1\xi_2 + 2\left[\frac{\rho_{12}-\rho_{13}\rho_{23}}{K\ \sigma_1\sigma_2}\right]^2$$

$$+ \left[\frac{1-\rho_{13}^2}{K\sigma_2^2}\right]\left[\frac{1-\rho_{23}^2}{K\sigma_1^2}\right]$$

$$H_{2,0,2} = \xi_1^2\xi_3^2 - \frac{(1-\rho_{12}^2)}{K\sigma_3^2}\ \xi_1^2 - \frac{(1-\rho_{23}^2)}{K\sigma_1^2}\ \xi_3^2 + 4\ \frac{(\rho_{13}-\rho_{12}\rho_{23})}{K\ \sigma_1\sigma_3}\ \xi_1\xi_3$$

$$+ 2\left[\frac{\rho_{13}-\rho_{12}\rho_{23}}{K\ \sigma_1\sigma_3}\right]^2 + \left[\frac{1-\rho_{12}^2}{K\sigma_3^2}\right]\left[\frac{1-\rho_{23}^2}{K\sigma_1^2}\right]$$

$$H_{0,2,2} = \xi_2^2\xi_3^2 - \frac{(1-\rho_{12}^2)}{K\ \sigma_3{}^2}\ \xi_2^2 - \frac{(1-\rho_{13}^2)}{K\sigma_2{}^2}\ \xi_3^2 + 4\ \frac{(\rho_{23}-\rho_{12}\rho_{13})}{K\ \sigma_2\sigma_3}\ \xi_2\xi_3$$

$$+ 2\left[\frac{\rho_{23}-\rho_{12}\rho_{13}}{K\ \sigma_2\sigma_3}\right]^2 + \left[\frac{1-\rho_{12}^2}{K\sigma_3{}^2}\right]\left[\frac{1-\rho_{13}^2}{K\sigma_2{}^2}\right] \qquad \text{(F.99 Cont.)}$$

$$H_{2,1,1} = \xi_1^2\xi_2\xi_3 + \frac{(\rho_{23}-\rho_{12}\rho_{13})}{K\ \sigma_2\sigma_3}\ \xi_1^2 + 2\ \frac{(\rho_{13}-\rho_{12}\rho_{23})}{K\ \sigma_1\sigma_3}\ \xi_1\xi_2$$

$$+ 2\ \frac{(\rho_{12}-\rho_{13}\rho_{23})}{K\ \sigma_1\sigma_2}\ \xi_1\xi_3 - \frac{(1-\rho_{23}^2)}{K\ \sigma_1^2}\ \xi_2\xi_3$$

$$+ 2\left[\frac{\rho_{12}-\rho_{13}\rho_{23}}{K\ \sigma_1\sigma_2}\right]\left[\frac{\rho_{13}-\rho_{12}\rho_{23}}{K\ \sigma_1\sigma_3}\right] - \left[\frac{1-\rho_{23}^2}{K\sigma_1{}^2}\right]\left[\frac{\rho_{23}-\rho_{12}\rho_{13}}{K\ \sigma_2\sigma_3}\right]$$

$$H_{1,2,1} = \xi_1\xi_2^2\xi_3 + \frac{(\rho_{13}-\rho_{12}\rho_{23})}{K\ \sigma_1\sigma_3}\ \xi_2^2 + 2\ \frac{(\rho_{12}-\rho_{13}\rho_{23})}{K\ \sigma_1\sigma_2}\ \xi_2\xi_3$$

$$+ 2\ \frac{(\rho_{23}-\rho_{12}\rho_{13})}{K\ \sigma_2\sigma_3}\ \xi_1\xi_2 - \frac{(1-\rho_{13}^2)}{K\ \sigma_2^2}\ \xi_1\xi_3$$

$$+ 2\left[\frac{\rho_{12}-\rho_{13}\rho_{23}}{K\ \sigma_1\sigma_2}\right]\left[\frac{\rho_{23}-\rho_{12}\rho_{13}}{K\ \sigma_2\sigma_3}\right] - \left[\frac{1-\rho_{13}^2}{K\sigma_2^2}\right]\left[\frac{\rho_{13}-\rho_{12}\rho_{23}}{K\ \sigma_1\sigma_3}\right]$$

(F.99 Cont.)

$$H_{1,1,2} = \xi_1\xi_2\xi_3^2 + \frac{(\rho_{12}-\rho_{13}\rho_{23})}{K\ \sigma_1\sigma_2}\ \xi_3^2 + 2\ \frac{(\rho_{23}-\rho_{12}\rho_{13})}{K\ \sigma_2\sigma_3}\ \xi_1\xi_3$$

$$+ 2\ \frac{(\rho_{13}-\rho_{12}\rho_{23})}{K\ \sigma_1\sigma_3}\ \xi_2\xi_3 - \frac{(1-\rho_{12}^2)}{K\ \sigma_3^2}\ \xi_1\xi_2$$

$$+ 2\left[\frac{\rho_{23}-\rho_{12}\rho_{13}}{K\ \sigma_2\sigma_3}\right]\left[\frac{\rho_{13}-\rho_{12}\rho_{23}}{K\ \sigma_1\sigma_3}\right] - \left[\frac{1-\rho_{12}^2}{K\sigma_3^2}\right]\left[\frac{\rho_{12}-\rho_{13}\rho_{23}}{K\ \sigma_1\sigma_2}\right]$$

$G_{0,0,0} = 1$ $\qquad G_{1,0,0} = v_1$ $\qquad G_{0,1,0} = v_2$ $\qquad G_{0,0,1} = v_3$

$$G_{1,1,0} = v_1v_2 - \rho_{12}\sigma_1\sigma_2$$

$$G_{1,0,1} = v_1v_3 - \rho_{13}\sigma_1\sigma_3 \qquad \text{(F.100)}$$

$$G_{0,1,1} = v_2v_3 - \rho_{23}\sigma_2\sigma_3$$

$$G_{2,0,0} = v_1^2 - \sigma_1^2$$

$$G_{0,2,0} = v_2^2 - \sigma_2^2$$

$$G_{0,0,2} = v_3^2 - \sigma_3^2$$

$$G_{3,0,0} = v_1^3 - 3\,\sigma_1^2 v_1$$

$$G_{0,3,0} = v_2^3 - 3\,\sigma_2^2 v_2$$

$$G_{0,0,3} = v_3^3 - 3\,\sigma_3^2 v_3$$

$$G_{2,1,0} = v_1^2 v_2 - 2\,\rho_{12}\sigma_1\sigma_2 v_1 - \sigma_1^2 v_2 \qquad \text{(F.100 Cont.)}$$

$$G_{1,2,0} = v_1 v_2^2 - 2\,\rho_{12}\sigma_1\sigma_2 v_2 - \sigma_2^2 v_1$$

$$G_{2,0,1} = v_1^2 v_3 - 2\,\rho_{13}\sigma_1\sigma_3 v_1 - \sigma_1^2 v_3$$

$$G_{1,0,2} = v_1 v_3^2 - 2\,\rho_{13}\sigma_1\sigma_3\, v_3 - \sigma_3^2 v_1$$

$$G_{0,2,1} = v_2^2 v_3 - 2\,\rho_{23}\sigma_2\sigma_3 v_2 - \sigma_2^2 v_3$$

$$G_{0,1,2} = v_2 v_3^2 - 2\,\rho_{23}\sigma_2\sigma_3 v_3 - \sigma_3^2 v_2$$

$$G_{1,1,1} = v_1 v_2 v_3 - \rho_{12}\sigma_1\sigma_2 v_3 - \rho_{13}\sigma_1\sigma_3 v_2 - \rho_{23}\sigma_2\sigma_3 v_1$$

$$G_{4,0,0} = v_1^{\,4} - 6\,\sigma_1^{\,2} v_1^{\,2} + 3\,\sigma_1^{\,4}$$

$$G_{0,4,0} = v_2^{\,4} - 6\,\sigma_2^{\,2} v_2^{\,2} + 3\,\sigma_2^{\,4}$$

$$G_{0,0,4} = v_3^{\,4} - 6\,\sigma_3^{\,2} v_3^{\,2} + 3\,\sigma_2^{\,4}$$

$$G_{3,1,0} = v_1^{\,3} v_2 - 3\,\rho_{12}\sigma_1\sigma_2 v_1^{\,2} - 3\,\sigma_1^{\,2} v_1 v_2 + 3\,\rho_{12}\sigma_1^{\,3}\sigma_2$$

$$G_{1,3,0} = v_1 v_2^{\,3} - 3\,\rho_{12}\sigma_1\sigma_2 v_2^{\,2} - 3\,\sigma_2^{\,2} v_1 v_2 + 3\,\rho_{12}\sigma_1\sigma_2^{\,3}$$

$$G_{3,0,1} = v_1^{\,3} v_3 - 3\,\rho_{13}\sigma_1\sigma_3 v_1^{\,2} - 3\,\sigma_1^{\,2} v_1 v_3 + 3\,\rho_{13}\sigma_1^{\,3}\sigma_3$$

$$G_{1,0,3} = v_1 v_3^{\,3} - 3\,\rho_{13}\sigma_1\sigma_3 v_3^{\,2} - 3\,\sigma_3^{\,2} v_1 v_3 + 3\,\rho_{13}\sigma_1\sigma_3^{\,3} \qquad \text{(F.100 Cont.)}$$

$$G_{0,3,1} = v_2^{\,3} v_3 - 3\,\rho_{23}\sigma_2\sigma_3 v_2^{\,2} - 3\,\sigma_2^{\,2} v_2 v_3 + 3\,\rho_{23}\sigma_2^{\,3}\sigma_3$$

$$G_{0,1,3} = v_2 v_3^{\,3} - 3\,\rho_{23}\sigma_2\sigma_3 v_3^{\,2} - 3\,\sigma_3^{\,2} v_2 v_3 + 3\,\rho_{23}\sigma_2\sigma_3^{\,3}$$

$$G_{2,2,0} = v_1^{\,2} v_2^{\,2} - \sigma_2^{\,2} v_1^{\,2} - \sigma_1^{\,2} v_2^{\,2} - 4\,\rho_{12}\sigma_1\sigma_2 v_1 v_2 + \sigma_1^{\,2}\sigma_2^{\,2}$$
$$+ 2(\rho_{12}\sigma_1\sigma_2)^2$$

$$G_{2,0,2} = v_1^{\,2} v_3^{\,2} - \sigma_3^{\,2} v_1^{\,2} - \sigma_1^{\,2} v_3^{\,2} - 4\,\rho_{13}\sigma_1\sigma_3 v_1 v_3 + \sigma_1^{\,2}\sigma_3^{\,2}$$
$$+ 2(\rho_{13}\sigma_1\sigma_3)^2$$

$$G_{0,2,2} = v_2^{\,2}v_3^{\,2} - \sigma_3^{\,2}v_2^{\,2} - \sigma_2^{\,2}v_3^{\,2} - 4\,\rho_{23}\sigma_2\sigma_3v_2v_3 + \sigma_2^{\,2}\sigma_3^{\,2}$$

$$+ 2(\rho_{23}\sigma_2\sigma_3)^2$$

$$G_{2,1,1} = v_1^{\,2}v_2v_3 - \rho_{23}\sigma_2\sigma_3v_1^{\,2} - 2\,\rho_{12}\sigma_1\sigma_2v_1v_3 - 2\,\rho_{13}\sigma_1\sigma_3v_1v_2$$

$$- \sigma_1^{\,2}v_2v_3 + \sigma_1^{\,2}\rho_{23}\sigma_2\sigma_3 + 2(\rho_{12}\sigma_1\sigma_2)(\rho_{13}\sigma_1\sigma_3)$$

$$G_{1,2,1} = v_1v_2^{\,2}v_3 - \rho_{13}\sigma_1\sigma_3v_2^{\,2} - 2\,\rho_{12}\sigma_1\sigma_2v_2v_3 - 2\,\rho_{23}\sigma_2\sigma_3v_1v_2$$

$$- \sigma_2^{\,2}v_1v_3 + \sigma_2^{\,2}\,\rho_{13}\sigma_1\sigma_3 + 2(\rho_{12}\sigma_1\sigma_2)(\rho_{23}\sigma_2\sigma_3)$$

$$G_{1,1,2} = v_1v_2v_3^{\,2} - \rho_{12}\sigma_1\sigma_2v_3^{\,2} - 2\,\rho_{13}\sigma_1\sigma_3v_2v_3 \qquad \text{(F.100 Cont.)}$$

$$- 2\,\rho_{23}\sigma_2\sigma_3v_1v_3 - \sigma_3^{\,2}v_1v_2$$

$$+ \sigma_3^{\,2}\rho_{12}\sigma_1\sigma_2 + 2(\rho_{13}\sigma_1\sigma_3)(\rho_{23}\sigma_2\sigma_3)$$

$$G_{500} = v_1^{\,5} - 10\,v_1^{\,3}\sigma_1^{\,2} + 15\,v_1\sigma_1^{\,4}$$

$$G_{050} = v_2^{\,5} - 10\,v_2^{\,3}\sigma_2^{\,2} + 15\,v_2\sigma_2^{\,4}$$

$$G_{005} = v_3^{\,5} - 10\,v_3^{\,3}\sigma_3^{\,2} + 15\,v_3\sigma_3^{\,4}$$

$$G_{410} = v_1^{\,4}v_2 - 4\,\rho_{12}\sigma_1\sigma_2v_1^{\,3} - 6\,\sigma_1^{\,2}v_1^{\,2}v_2$$

$$+ 12\ \rho_{12}\sigma_1^{\ 3}\sigma_2 v_1 + 3\ \sigma_1^{\ 4} v_2$$

$$G_{401} = v_1^{\ 4} v_3 - 4\ \rho_{13}\sigma_1\sigma_3 v_1^{\ 3} - 6\ \sigma_1^{\ 2} v_1^{\ 2} v_3$$

$$+ 12\ \rho_{13}\sigma_1^{\ 3}\sigma_3 v_1 + 3\ \sigma_1^{\ 4} v_3$$

$$G_{140} = v_1 v_2^{\ 4} - 4\ \rho_{12}\sigma_1\sigma_2 v_2^{\ 3} - 6\ \sigma_2^{\ 2} v_1 v_2^{\ 2}$$

$$+ 12\ \rho_{12}\sigma_1\sigma_2^{\ 3} v_2 + 3\ \sigma_2^{\ 4} v_1$$

$$G_{104} = v_1 v_3^{\ 4} - 4\ \rho_{13}\sigma_1\sigma_3 v_3^{\ 3} - 6\ \sigma_3^{\ 2} v_1 v_3^{\ 2}$$

$$+ 12\ \rho_{13}\sigma_1\sigma_3^{\ 3} v_3 + 3\ \sigma_3^{\ 4} v_1 \qquad \text{(F.100 Cont.)}$$

$$G_{041} = v_2^{\ 4} v_3 - 4\ \rho_{23}\sigma_2\sigma_3\ v_2^{\ 3} - 6\ \sigma_2^{\ 2} v_2^{\ 2} v_3$$

$$+ 12\ \rho_{23}\sigma_2^{\ 3}\sigma_3 v_2 + 3\ \sigma_2^{\ 4} v_3$$

$$G_{014} = v_2 v_3^{\ 4} - 4\ \rho_{23}\sigma_2\sigma_3 v_3^{\ 3} - 6\ \sigma_3^{\ 2} v_2 v_3^{\ 2}$$

$$+ 12\ \rho_{23}\sigma_2\sigma_3^{\ 3} v_3 + 3\ \sigma_3^{\ 4} v_2$$

$$G_{320} = v_1^{\ 3} v_2^{\ 2} - \sigma_2^{\ 2} v_1^{\ 3} - 3\ \sigma_1^{\ 2} v_1 v_2^{\ 2}$$

$$- 6\ \rho_{12}\sigma_1\sigma_2 v_1^{\ 2} v_2 + [3\ \sigma_1^{\ 2}\sigma_2^{\ 2} + 6(\rho_{12}\sigma_1\sigma_2)^2] v_1$$

$$+ 6\ \rho_{12}\sigma_1^{\ 3}\sigma_2 v_2$$

$$G_{230} = v_1^{\ 2} v_2^{\ 3} - \sigma_1^{\ 2} v_2^{\ 3} - 3\ \sigma_2^{\ 2} v_1^{\ 2} v_2 - 6\ \rho_{12}\sigma_1\sigma_2 v_1 v_2^{\ 2}$$

$$+ [3\ \sigma_1^{\ 2}\sigma_2^{\ 2} + 6(\rho_{12}\sigma_1\sigma_2)^2]\ v_2 + 6\ \rho_{12}\sigma_1\sigma_2^{\ 3} v_1$$

$$G_{302} = v_1^{\ 3} v_3^{\ 2} - \sigma_3^{\ 2} v_1^{\ 3} - 3\ \sigma_1^{\ 2}\ v_1 v_3^{\ 2} - 6\ \rho_{13}\sigma_1\sigma_3 v_1^{\ 2} v_3$$

$$+ [3\ \sigma_1^{\ 2}\sigma_3^{\ 2} + 6(\rho_{13}\sigma_1\sigma_3)^2]\ v_1 + 6\ \rho_{13}\sigma_1^{\ 3}\sigma_3 v_3$$

$$G_{203} = v_1^{\ 2} v_3^{\ 3} - \sigma_1^{\ 2} v_3^{\ 3} - 3\ \sigma_3^{\ 2} v_1^{\ 2} v_3 - 6\ \rho_{13}\sigma_1\sigma_3 v_1 v_3^{\ 2}$$

$$+ [3\ \sigma_1^{\ 2}\sigma_3^{\ 2} + 6(\rho_{13}\sigma_1\sigma_3)^2]\ v_3 + 6\ \rho_{13}\sigma_1\sigma_3^{\ 3} v_1 \qquad \text{(F.100 Cont.)}$$

$$G_{032} = v_2^{\ 3} v_3^{\ 2} - \sigma_3^{\ 2} v_2^{\ 3} - 3\ \sigma_2^{\ 2} v_2 v_3^{\ 2} - 6\ \rho_{23}\sigma_2\sigma_3 v_2^{\ 2} v_3$$

$$+ [3\ \sigma_2^{\ 2}\sigma_3^{\ 2} + 6(\rho_{23}\sigma_2\sigma_3)^2]\ v_2 + 6\ \rho_{23}\sigma_2^{\ 3}\sigma_3 v_3$$

$$G_{023} = v_2^{\ 2} v_3^{\ 3} - \sigma_2^{\ 2} v_3^{\ 3} - 3\ \sigma_3^{\ 2} v_2^{\ 2} v_3 - 6\ \rho_{23}\sigma_2\sigma_3 v_2 v_3^{\ 2}$$

$$+ [3\ \sigma_2^{\ 2}\sigma_3^{\ 2} + 6(\rho_{23}\sigma_2\sigma_3)^2]\ v_3 + 6\ \rho_{23}\sigma_2\sigma_3^{\ 3} v_2$$

$$G_{311} = v_1^{\ 3} v_2 v_3 - \rho_{23}\sigma_2\sigma_3 v_1^{\ 3} - 3\ \rho_{12}\sigma_1\sigma_2 v_1^{\ 2} v_3$$

$$- 3\ \rho_{13}\sigma_1\sigma_3 v_1^{\ 2} v_2 - 3\ \sigma_1^{\ 2} v_1 v_2 v_3 + [3\ \sigma_1^{\ 2}\rho_{23}\sigma_2\sigma_3$$

$$+ 6(\rho_{12}\sigma_1\sigma_2)(\rho_{13}\sigma_1\sigma_3)]\ v_1 + 3\ \sigma_1^{\ 2}\rho_{12}\sigma_1\sigma_2 v_3 + 3\sigma_1^{\ 2}\rho_{13}\sigma_1\sigma_3 v_2$$

$$G_{131} = v_1 v_2^{\ 3} v_3 - \rho_{13}\sigma_1\sigma_3 v_2^{\ 3} - 3\ \rho_{12}\sigma_1\sigma_2 v_2^{\ 2} v_3$$

$$- 3\ \rho_{23}\sigma_2\sigma_3 v_1 v_2^{\ 2} - 3\ \sigma_2^{\ 2} v_1 v_2 v_3 + [3\ \sigma_2^{\ 2}\rho_{13}\sigma_1\sigma_3$$

$$+ 6(\rho_{12}\sigma_1\sigma_2)(\rho_{23}\sigma_2\sigma_3)]\ v_2 + 3\ \sigma_2^{\ 2}\ \rho_{12}\sigma_1\sigma_2 v_3 + 3\sigma_2^{\ 2}\rho_{23}\sigma_2\sigma_3 v_1$$

$$G_{113} = v_1 v_2 v_3^{\ 3} - \rho_{12}\sigma_1\sigma_2 v_3^{\ 3} - 3\ \rho_{13}\sigma_1\sigma_3 v_2 v_3^{\ 2}$$

$$- 3\ \rho_{23}\sigma_2\sigma_3 v_1 v_3^{\ 2} - 3\ \sigma_3^{\ 2} v_1 v_2 v_3 + [3\ \sigma_3^{\ 2}\ \rho_{12}\sigma_1\sigma_2$$

$$+ 6(\rho_{13}\sigma_1\sigma_3)(\rho_{23}\sigma_2\sigma_3)]\ v_3 + 3\ \sigma_3^{\ 2}\ \rho_{13}\sigma_1\sigma_3 v_2 + 3\sigma_3^{\ 2}\rho_{23}\sigma_2\sigma_3 v_1$$

(F.100 Cont.)

$$G_{221} = v_1^{\ 2} v_2^{\ 2} v_3 - 2\ \rho_{13}\sigma_1\sigma_3 v_1 v_2^{\ 2} - 2\ \rho_{23}\sigma_2\sigma_3 v_1^{\ 2} v_2 - \sigma_2^{\ 2} v_1^{\ 2} v_3$$

$$- \sigma_1^{\ 2} v_2^{\ 2} v_3 - 4\ \rho_{12}\sigma_1\sigma_2 v_1 v_2 v_3 + [2\ \sigma_2^{\ 2}\ \rho_{13}\sigma_1\sigma_3$$

$$+ 4(\rho_{12}\sigma_1\sigma_2)(\rho_{23}\sigma_2\sigma_3)]v_1 + [2\ \sigma_1^{\ 2}\ \rho_{23}\sigma_2\sigma_3$$

$$+ 4(\rho_{12}\sigma_1\sigma_2)(\rho_{13}\sigma_1\sigma_3)]\ v_2 + [\sigma_1^{\ 2}\sigma_2^{\ 2} + 2(\rho_{12}\sigma_1\sigma_2)^2]v_3$$

$$G_{212} = v_1^{\ 2} v_2 v_3^{\ 2} - 2\ \rho_{12}\sigma_1\sigma_2 v_1 v_3^{\ 2} - 2\ \rho_{23}\sigma_2\sigma_3 v_1^{\ 2} v_3 - \sigma_3^{\ 2} v_1^{\ 2} v_2$$

$$- \sigma_1^{\ 2} v_2 v_3^{\ 2} - 4\ \rho_{13}\sigma_1\sigma_3 v_1 v_2 v_3 + [2\ \sigma_3^{\ 2}\rho_{12}\sigma_1\sigma_2$$

$$+ 4(\rho_{13}\sigma_1\sigma_3)(\rho_{23}\sigma_2\sigma_3)]v_1 + [2\ \sigma_1^{2}\rho_{23}\sigma_2\sigma_3$$

$$+ 4(\rho_{13}\sigma_1\sigma_3)(\rho_{12}\sigma_1\sigma_2)]v_3 + [\sigma_1^{2}\sigma_3^{2} + 2(\rho_{13}\sigma_1\sigma_3)^{2}]v_2$$

$$G_{122} = v_1v_2^{2}v_3^{2} - 2\ \rho_{12}\sigma_1\sigma_2v_2v_3^{2} - 2\ \rho_{13}\sigma_1\sigma_3v_2^{2}v_3 - \sigma_3^{2}\ v_1v_2^{2}$$

$$- \sigma_2^{2}v_1v_3^{2} - 4\ \rho_{23}\sigma_2\sigma_3v_1v_2v_3 + [2\ \sigma_3^{2}\rho_{12}\sigma_1\sigma_2$$

$$+ 4\ (\rho_{13}\sigma_1\sigma_3)(\rho_{23}\sigma_2\sigma_3)]v_2 + [2\ \sigma_2^{2}\rho_{13}\sigma_1\sigma_3$$

$$+ 4(\rho_{12}\sigma_1\sigma_2)(\rho_{23}\sigma_2\sigma_3)]v_3 + [\sigma_2^{2}\sigma_3^{2} + 2(\rho_{23}\sigma_2\sigma_3)^{2}]v_1$$

$$G_{600} = v_1^{6} - 15\ v_1^{4}\sigma_1^{2} + 45\ v_1^{2}\sigma_1^{4} - 15\ \sigma_1^{6} \qquad \text{(F.100 Cont.)}$$

$$G_{060} = v_2^{6} - 15\ v_2^{4}\sigma_2^{2} + 45\ v_2^{2}\sigma_2^{4} - 15\ \sigma_2^{6}$$

$$G_{006} = v_3^{6} - 15\ v_3^{4}\sigma_3^{2} + 45\ v_3^{2}\sigma_3^{4} - 15\ \sigma_3^{6}$$

$$G_{510} = v_1^{5}v_2 - 5\ \rho_{12}\sigma_1\sigma_2v_1^{4} - 10\ \sigma_1^{2}v_1^{3}v_2 + 30\ \rho_{12}\sigma_1^{3}\sigma_2v_1^{2}$$

$$+ 15\ \sigma_1^{4}v_1v_2 - 15\ \rho_{12}\sigma_1^{5}\sigma_2$$

$$G_{150} = v_1v_2^{5} - 5\ \rho_{12}\sigma_1\sigma_2v_2^{4} - 10\ \sigma_2^{2}v_1v_2^{3} + 30\ \rho_{12}\sigma_1\sigma_2^{3}v_2^{2}$$

$$+ 15\ \sigma_2^{4}v_1v_2 - 15\ \rho_{12}\sigma_1\sigma_2^{5}$$

$$G_{501} = v_1^{5}v_3 - 5\ \rho_{13}\sigma_1\sigma_3 v_1^{4} - 10\ \sigma_1^{2}v_1^{3}v_3 + 30\ \rho_{13}\ \sigma_1^{3}\sigma_3 v_1^{2}$$

$$+ 15\ \sigma_1^{4}v_1v_3 - 15\ \rho_{13}\sigma_1^{5}\sigma_3$$

$$G_{105} = v_1v_3^{5} - 5\ \rho_{13}\sigma_1\sigma_3 v_3^{4} - 10\ \sigma_3^{2}v_1v_3^{3} + 30\ \rho_{13}\sigma_1\sigma_3^{3}v_3^{2}$$

$$+ 15\ \sigma_3^{4}v_1v_3 - 15\ \rho_{13}\sigma_1\sigma_3^{5}$$

$$G_{051} = v_2^{5}v_3 - 5\ \rho_{23}\sigma_2\sigma_3 v_2^{4} - 10\ \sigma_2^{2}v_2^{3}v_3 + 30\ \rho_{23}\sigma_2^{3}\sigma_3 v_2^{2}$$

$$+ 15\ \sigma_2^{4}v_2v_3 - 15\ \rho_{23}\sigma_2^{5}\sigma_3$$

$$G_{015} = v_2v_3^{5} - 5\ \rho_{23}\sigma_2\sigma_3 v_3^{4} - 10\ \sigma_3^{2}v_2v_3^{3} + 30\ \rho_{23}\sigma_2\sigma_3^{3}v_3^{2} \qquad \text{(F.100 Cont.)}$$

$$+ 15\ \sigma_3^{4}v_2v_3 - 15\ \rho_{23}\sigma_2\sigma_3^{5}$$

$$G_{420} = v_1^{4}v_2^{2} - \sigma_2^{2}v_1^{4} - 6\ \sigma_1^{2}v_1^{2}v_2^{2} - 8\ \rho_{12}\sigma_1\sigma_2 v_1^{3}v_2$$

$$+ [6\ \sigma_1^{2}\sigma_2^{2} + 12(\rho_{12}\sigma_1\sigma_2)^{2}]\ v_1^{2} + 24\ \rho_{12}\sigma_1^{3}\sigma_2 v_1v_2$$

$$+ 3\ \sigma_1^{4}v_2^{2} - [3\ \sigma_1^{4}\sigma_2^{2} + 12\ \sigma_1^{2}\ (\rho_{12}\sigma_1\sigma_2)^{2}]$$

$$G_{240} = v_1^{2}v_2^{4} - \sigma_1^{2}v_2^{4} - 6\ \sigma_2^{2}v_1^{2}v_2^{2} - 8\ \rho_{12}\sigma_1\sigma_2 v_1v_2^{3}$$

$$+ [6\ \sigma_1^{2}\sigma_2^{2} + 12(\rho_{12}\sigma_1\sigma_2)^{2}]v_2^{2} + 24\ \rho_{12}\sigma_1\sigma_2^{3}v_1v_2$$

$$+ 3\ \sigma_2^{\ 4} v_1^{\ 2} - [3\ \sigma_1^{\ 2}\sigma_2^{\ 4} + 12\ \sigma_2^{\ 2}(\rho_{12}\sigma_1\sigma_2)^2]$$

$$G_{402} = v_1^{\ 4} v_3^{\ 2} - \sigma_3^{\ 2} v_1^{\ 4} - 6\ \sigma_1^{\ 2} v_1^{\ 2} v_3^{\ 2} - 8\ \rho_{13}\sigma_1\sigma_3 v_1^{\ 3} v_3$$

$$+ [6\ \sigma_1^{\ 2}\sigma_3^{\ 2} + 12\ (\rho_{13}\sigma_1\sigma_3)^2] v_1^{\ 2} + 24\ \rho_{13}\sigma_1^{\ 3}\sigma_3 v_1 v_3$$

$$+ 3\ \sigma_1^{\ 4} v_3^{\ 2} - [3\ \sigma_1^{\ 4}\sigma_3^{\ 2} + 12\ \sigma_1^{\ 2}(\rho_{13}\sigma_1\sigma_3)^2]$$

$$G_{204} = v_1^{\ 2} v_3^{\ 4} - \sigma_1^{\ 2} v_3^{\ 4} - 6\ \sigma_3^{\ 2} v_1^{\ 2} v_3^{\ 2} - 8\ \rho_{13}\sigma_1\sigma_3 v_1 v_3^{\ 3}$$

$$+ [6\ \sigma_1^{\ 2}\sigma_3^{\ 2} + 12(\rho_{13}\sigma_1\sigma_3)^2]\ v_3^{\ 2} + 24\ \rho_{13}\sigma_1\sigma_3^{\ 3} v_1 v_3$$

$$+ 3\ \sigma_3^{\ 4} v_1^{\ 2} - [3\ \sigma_1^{\ 2}\sigma_3^{\ 4} + 12\ \sigma_3^{\ 2}(\rho_{13}\sigma_1\sigma_3)^2] \qquad \text{(F.100 Cont.)}$$

$$G_{042} = v_2^{\ 4} v_3^{\ 2} - \sigma_3^{\ 2} v_2^{\ 4} - 6\ \sigma_2^{\ 2} v_2^{\ 2} v_3^{\ 2} - 8\ \rho_{23}\sigma_2\sigma_3 v_2^{\ 3} v_3$$

$$+ [6\ \sigma_2^{\ 2}\sigma_3^{\ 2} + 12(\rho_{23}\sigma_2\sigma_3)^2] v_2^{\ 2} + 24\ \rho_{23}\sigma_2^{\ 3}\sigma_3 v_2 v_3$$

$$+ 3\ \sigma_2^{\ 4} v_3^{\ 2} - [3\ \sigma_2^{\ 4}\sigma_3^{\ 2} + 12\ \sigma_2^{\ 2}(\rho_{23}\sigma_2\sigma_3)^2]$$

$$G_{024} = v_2^{\ 2} v_3^{\ 4} - \sigma_2^{\ 2} v_3^{\ 4} - 6\ \sigma_3^{\ 2} v_2^{\ 2} v_3^{\ 2} - 8\ \rho_{23}\sigma_2\sigma_3 v_2 v_3^{\ 3}$$

$$+ [6\ \sigma_2^{\ 2}\sigma_3^{\ 2} + 12(\rho_{23}\sigma_2\sigma_3)^2] v_3^{\ 2} + 24\ \rho_{23}\sigma_2\sigma_3^{\ 3} v_2 v_3$$

$$+ 3\ \sigma_3^{\ 4} v_2^{\ 2} - [3\ \sigma_2^{\ 2}\sigma_3^{\ 4} + 12\ \sigma_3^{\ 2}(\rho_{23}\sigma_2\sigma_3)^2]$$

$$G_{411} = v_1^{\,4} v_2 v_3 - \rho_{23}\sigma_2\sigma_3 v_1^{\,4} - 4\,\rho_{12}\sigma_1\sigma_2 v_1^{\,3} v_3 - 4\,\rho_{13}\sigma_1\sigma_3 v_1^{\,3} v_2$$

$$- 6\,\sigma_1^{\,2} v_1^{\,2} v_2 v_3 + [6\,\sigma_1^{\,2}\rho_{23}\sigma_2\sigma_3 + 12(\rho_{12}\sigma_1\sigma_2)(\rho_{13}\sigma_1\sigma_3)]\,v_1^{\,2}$$

$$+ 12\,\sigma_1^{\,2}\rho_{13}\sigma_1\sigma_3 v_1 v_2 + 12\,\sigma_1^{\,2}\rho_{12}\sigma_1\sigma_2 v_1 v_3 + 3\,\sigma_1^{\,4} v_2 v_3$$

$$- [3\,\sigma_1^{\,4}\,\rho_{23}\sigma_2\sigma_3 + 12\,\sigma_1^{\,2}(\rho_{12}\sigma_1\sigma_2)(\rho_{13}\sigma_1\sigma_3)]$$

$$G_{141} = v_1 v_2^{\,4} v_3 - \rho_{13}\sigma_1\sigma_3 v_2^{\,4} - 4\,\rho_{12}\sigma_1\sigma_2 v_2^{\,3} v_3 - 4\,\rho_{23}\sigma_2\sigma_3 v_1 v_2^{\,3}$$

$$- 6\,\sigma_2^{\,2} v_1 v_2^{\,2} v_3 + [6\,\sigma_2^{\,2}\rho_{13}\sigma_1\sigma_3 + 12(\rho_{12}\sigma_1\sigma_2)(\rho_{23}\sigma_2\sigma_3)]v_2^{\,2}$$

$$+ 12\,\sigma_2^{\,2}\,\rho_{23}\sigma_2\sigma_3 v_1 v_2 + 12\,\sigma_2^{\,2}\,\rho_{12}\sigma_1\sigma_2 v_2 v_3 + 3\,\sigma_2^{\,4} v_1 v_3 \qquad \text{(F.100 Cont.)}$$

$$- [3\,\sigma_2^{\,4}\rho_{13}\sigma_1\sigma_3 + 12\,\sigma_2^{\,2}(\rho_{12}\sigma_1\sigma_2)(\rho_{23}\sigma_2\sigma_3)]$$

$$G_{114} = v_1 v_2 v_3^{\,4} - \rho_{12}\sigma_1\sigma_2 v_3^{\,4} - 4\,\rho_{13}\sigma_1\sigma_3 v_2 v_3^{\,3} - 4\,\rho_{23}\sigma_2\sigma_3 v_1 v_3^{\,3}$$

$$- 6\,\sigma_3^{\,2}\,v_1 v_2 v_3^{\,2} + [6\,\sigma_3^{\,2}\,\rho_{12}\sigma_1\sigma_2 + 12(\rho_{13}\sigma_1\sigma_3)(\rho_{23}\sigma_2\sigma_3)]v_3^{\,2}$$

$$+ 12\,\sigma_3^{\,2}\,\rho_{23}\sigma_2\sigma_3 v_1 v_3 + 12\,\sigma_3^{\,2}\,\rho_{13}\sigma_1\sigma_3 v_2 v_3 + 3\,\sigma_3^{\,4} v_1 v_2$$

$$- [3\,\sigma_3^{\,4}\,\rho_{12}\sigma_1\sigma_2 + 12\,\sigma_3^{\,2}(\rho_{13}\sigma_1\sigma_3)(\rho_{23}\sigma_2\sigma_3)]$$

$$G_{330} = v_1^{\,3} v_2^{\,3} - 3\,\sigma_1^{\,2} v_1 v_2^{\,3} - 3\,\sigma_2^{\,2} v_1^{\,3} v_2 - 9\,\rho_{12}\sigma_1\sigma_2 v_1^{\,2} v_2^{\,2}$$

$$+ [9\ \sigma_1{}^2\sigma_2{}^2 + 18(\rho_{12}\sigma_1\sigma_2)^2]\ v_1v_2 + 9\ \rho_{12}\sigma_1\sigma_2{}^3v_1{}^2$$

$$+ [9\ \rho_{12}\sigma_1{}^3\sigma_2{}^3 + 6(\rho_{12}\sigma_1\sigma_2)^3] + 9\ \rho_{12}\sigma_1{}^3\sigma_2v_2{}^2$$

$$G_{303} = v_1{}^3v_3{}^3 - 3\ \sigma_1{}^2v_1v_3{}^3 - 3\ \sigma_3{}^2v_1{}^3v_3 - 9\ \rho_{13}\sigma_1\sigma_3v_1{}^2v_3{}^2$$

$$+ [9\ \sigma_1{}^2\sigma_3{}^2 + 18(\rho_{13}\sigma_1\sigma_3)^2]v_1v_3 + 9\ \rho_{13}\sigma_1\sigma_3{}^3v_1{}^2$$

$$+ 9\ \rho_{13}\sigma_1{}^3\sigma_3v_3{}^2 - [9\ \rho_{13}\sigma_1{}^3\sigma_3{}^3 + 6(\rho_{13}\sigma_1\sigma_3)^3]$$

$$G_{033} = v_2{}^3v_3{}^3 - 3\ \sigma_2{}^2v_2v_3{}^3 - 3\ \sigma_3{}^2v_2{}^3v_3 - 9\ \rho_{23}\sigma_2\sigma_3v_2{}^2v_3{}^2$$

$$+ [9\ \sigma_2{}^2\sigma_3{}^2 + 18(\rho_{23}\sigma_2\sigma_3)^2]v_2v_3 + 9\ \rho_{23}\sigma_2\sigma_3{}^3v_2{}^2 \qquad \text{(F.100 Cont.)}$$

$$+ 9\ \rho_{23}\sigma_2{}^3\sigma_3v_3{}^2 - [9\ \rho_{23}\sigma_2{}^3\sigma_3{}^3 + 6(\rho_{23}\sigma_2\sigma_3)^3]$$

$$G_{321} = v_1{}^3v_2{}^2v_3 - 3\ \rho_{13}\sigma_1\sigma_3v_1{}^2v_2{}^2 - 2\ \rho_{23}\sigma_2\sigma_3v_1{}^3v_2 - \sigma_2{}^2v_1{}^3v_3$$

$$- 6\ \rho_{12}\sigma_1\sigma_2v_1{}^2v_2v_3 - 3\ \sigma_1{}^2v_1v_2{}^2v_3 + [3\ \sigma_2{}^2\ \rho_{13}\sigma_1\sigma_3$$

$$+ 6(\rho_{12}\sigma_1\sigma_2)(\rho_{23}\sigma_2\sigma_3)]\ v_1{}^2 + 3\ \sigma_1{}^2\rho_{13}\sigma_1\sigma_3v_2{}^2$$

$$+ [6\ \sigma_1{}^2\rho_{23}\sigma_2\sigma_3 + 12(\rho_{12}\sigma_1\sigma_2)(\rho_{13}\sigma_1\sigma_3)]v_1v_2$$

$$+ [3\ \sigma_1{}^2\sigma_2{}^2 + 6(\rho_{12}\sigma_1\sigma_2)^2]v_1v_3 + 6\ \sigma_1{}^2\ \rho_{12}\sigma_1\sigma_2v_2v_3$$

$$- [3\ \sigma_1{}^2\sigma_2{}^2\rho_{13}\sigma_1\sigma_3 + 6(\rho_{13}\sigma_1\sigma_3)(\rho_{12}\sigma_1\sigma_2)^2$$

$$+ 6\ \sigma_1{}^2(\rho_{12}\sigma_1\sigma_2)(\rho_{23}\sigma_2\sigma_3)]$$

$$G_{231} = v_1{}^2 v_2{}^3 v_3 - 3\ \rho_{23}\sigma_2\sigma_3 v_1{}^2 v_2{}^2 - 2\ \rho_{13}\sigma_1\sigma_3 v_1 v_2{}^3 - \sigma_1{}^2 v_2{}^3 v_3$$

$$- 6\ \rho_{12}\sigma_1\sigma_2 v_1 v_2{}^2 v_3 - 3\ \sigma_2{}^2 v_1{}^2 v_2 v_3 + [3\ \sigma_1{}^2\rho_{23}\sigma_2\sigma_3$$

$$+ 6(\rho_{12}\sigma_1\sigma_2)(\rho_{13}\sigma_1\sigma_3)]v_2{}^2 + 3\ \sigma_2{}^2\ \rho_{23}\sigma_2\sigma_3 v_1{}^2$$

$$+ [6\ \sigma_2{}^2\rho_{13}\sigma_1\sigma_3 + 12(\rho_{12}\sigma_1\sigma_2)(\rho_{23}\sigma_2\sigma_3)]v_1 v_2$$

$$+ [3\ \sigma_1{}^2\sigma_2{}^2 + 6(\rho_{12}\sigma_1\sigma_2)^2]\ v_2 v_3 + 6\ \sigma_2{}^2\rho_{12}\sigma_1\sigma_2 v_1 v_3 \qquad \text{(F.100 Cont.)}$$

$$-[3\ \sigma_1{}^2\sigma_2{}^2\rho_{23}\sigma_2\sigma_3 + 6(\rho_{23}\sigma_2\sigma_3)(\rho_{12}\sigma_1\sigma_2)^2$$

$$+ 6\ \sigma_2{}^2(\rho_{12}\sigma_1\sigma_2)(\rho_{13}\sigma_1\sigma_3)]$$

$$G_{312} = v_1{}^3 v_2 v_3{}^2 - 3\ \rho_{12}\sigma_1\sigma_2 v_1{}^2 v_3{}^2 - 2\ \rho_{23}\sigma_2\sigma_3 v_1{}^3 v_3$$

$$- \sigma_3{}^2 v_1{}^3 v_2 - 6\ \rho_{13}\sigma_1\sigma_3 v_1{}^2 v_2 v_3 - 3\ \sigma_1{}^2 v_1 v_2 v_3{}^2$$

$$+ [3\ \sigma_3{}^2\rho_{12}\sigma_1\sigma_2 + 6(\rho_{13}\sigma_1\sigma_3)(\rho_{23}\sigma_2\sigma_3)]v_1{}^2$$

$$+ 3\ \sigma_1{}^2\rho_{12}\sigma_1\sigma_2 v_3{}^2 + [6\ \sigma_1{}^2\rho_{23}\sigma_2\sigma_3$$

$$+ 12(\rho_{13}\sigma_1\sigma_3)(\rho_{12}\sigma_1\sigma_2)]v_1v_3 + [3\ \sigma_1^{\ 2}\sigma_3^{\ 2}$$

$$+ 6(\rho_{13}\sigma_1\sigma_3)^2]v_1v_2 + 6\ \sigma_1^{\ 2}\rho_{13}\sigma_1\sigma_3v_2v_3$$

$$- [3\ \sigma_1^{\ 2}\sigma_3^{\ 2}\rho_{12}\sigma_1\sigma_2 + 6(\rho_{12}\sigma_1\sigma_2)(\rho_{13}\sigma_1\sigma_3)^2$$

$$+ 6\ \sigma_1^{\ 2}(\rho_{13}\sigma_1\sigma_3)(\rho_{23}\sigma_2\sigma_3)]$$

$$G_{213} = v_1^{\ 2}v_2v_3^{\ 3} - 3\ \rho_{23}\sigma_2\sigma_3v_1^{\ 2}v_3^{\ 2} - 2\ \rho_{12}\sigma_1\sigma_2v_1v_3^{\ 3} - \sigma_1^{\ 2}v_2v_3^{\ 3}$$

$$- 6\ \rho_{13}\sigma_1\sigma_3v_1v_2v_3^{\ 2} - 3\ \sigma_3^{\ 2}v_1^{\ 2}v_2v_3 + [3\ \sigma_1^{\ 2}\rho_{23}\sigma_2\sigma_3$$

$$+ 6(\rho_{13}\sigma_1\sigma_3)(\rho_{12}\sigma_1\sigma_2)]v_3^{\ 2} + 3\ \sigma_3^{\ 2}\rho_{23}\sigma_2\sigma_3v_1^{\ 2} \qquad \text{(F.100 Cont.)}$$

$$+ [6\sigma_3^{\ 2}\rho_{12}\sigma_1\sigma_2 + 12(\rho_{13}\sigma_1\sigma_3)(\rho_{23}\sigma_2\sigma_3)]v_1v_3$$

$$+ [3\sigma_1^{\ 2}\sigma_3^{\ 2} + 6(\rho_{13}\sigma_1\sigma_3)^2]v_2v_3 + 6\ \sigma_3^{\ 2}\rho_{13}\sigma_1\sigma_3v_1v_2$$

$$- [3\ \sigma_1^{\ 2}\sigma_3^{\ 2}\rho_{23}\sigma_2\sigma_3 + 6(\rho_{23}\sigma_2\sigma_3)(\rho_{13}\sigma_1\sigma_3)^2$$

$$+ 6\ \sigma_3^{\ 2}(\rho_{13}\sigma_1\sigma_3)(\rho_{12}\sigma_1\sigma_2)]$$

$$G_{132} = v_1v_2^{\ 3}v_3^{\ 2} - 3\ \rho_{12}\sigma_1\sigma_2v_2^{\ 2}v_3^{\ 2} - 2\ \rho_{13}\sigma_1\sigma_3v_2^{\ 3}v_3$$

$$- \sigma_3^{\ 2}v_1v_2^{\ 3} - 6\ \rho_{23}\sigma_2\sigma_3v_1v_2^{\ 2}v_3 - 3\ \sigma_2^{\ 2}v_1v_2v_3^{\ 2}$$

$$+ [3\ \sigma_3^{\ 2}\ \rho_{12}\sigma_1\sigma_2 + 6(\rho_{23}\sigma_2\sigma_3)(\rho_{13}\sigma_1\sigma_3)]v_2^{\ 2}$$

$$+ 3\ \sigma_2^{\ 2}\ \rho_{12}\sigma_1\sigma_2 v_3^{\ 2} + [6\ \sigma_2^{\ 2}\ \rho_{13}\sigma_1\sigma_3$$

$$+ 12(\rho_{23}\sigma_2\sigma_3)(\rho_{12}\sigma_1\sigma_2)]\ v_2 v_3 + [3\ \sigma_2^{\ 2}\sigma_3^{\ 2}$$

$$+ 6(\rho_{23}\sigma_2\sigma_3)^2]v_1 v_2 + 6\ \sigma_2^{\ 2}\ \rho_{23}\sigma_2\sigma_3 v_1 v_3$$

$$- [3\ \sigma_2^{\ 2}\sigma_3^{\ 2}\ \rho_{12}\sigma_1\sigma_2 + 6(\rho_{12}\sigma_1\sigma_2)(\rho_{23}\sigma_2\sigma_3)^2$$

$$+ 6\ \sigma_2^{\ 2}(\rho_{23}\sigma_2\sigma_3)(\rho_{13}\sigma_1\sigma_3)]$$

$$G_{123} = v_1 v_2^{\ 2} v_3^{\ 3} - 3\ \rho_{13}\sigma_1\sigma_3 v_2^{\ 2} v_3^{\ 2} - 2\ \rho_{12}\sigma_1\sigma_2 v_2 v_3^{\ 3} \qquad \text{(F.100 Cont.)}$$

$$- \sigma_2^{\ 2} v_1 v_3^{\ 3} - 6\ \rho_{23}\sigma_2\sigma_3 v_1 v_2 v_3^{\ 2} - 3\ \sigma_3^{\ 2} v_1 v_2^{\ 2} v_3$$

$$+ [3\ \sigma_2^{\ 2}\ \rho_{13}\sigma_1\sigma_3 + 6(\rho_{23}\sigma_2\sigma_3)(\rho_{12}\sigma_1\sigma_2)]v_3^{\ 2}$$

$$+ 3\sigma_3^{\ 2}\rho_{13}\sigma_1\sigma_3 v_2^{\ 2} + [6\ \sigma_3^{\ 2}\rho_{12}\sigma_1\sigma_2 + 12(\rho_{23}\sigma_2\sigma_3)(\rho_{13}\sigma_1\sigma_3)]v_2 v_3$$

$$+ [3\sigma_2^{\ 2}\sigma_3^{\ 2} + 6(\rho_{23}\sigma_2\sigma_3)^2]v_1 v_3 + 6\ \sigma_3^{\ 2}\rho_{23}\sigma_2\sigma_3 v_1 v_2$$

$$- [3\ \sigma_3^{\ 2}\sigma_2^{\ 2}\rho_{13}\sigma_1\sigma_3 + 6(\rho_{13}\sigma_1\sigma_3)(\rho_{23}\sigma_2\sigma_2)^2$$

$$+ 6\ \sigma_3^{\ 2}(\rho_{23}\sigma_2\sigma_3)(\rho_{12}\sigma_1\sigma_2)]$$

$$G_{222} = v_1^2 v_2^2 v_3^2 - 4\,\rho_{12}\sigma_1\sigma_2 v_1 v_2 v_3^2 - 4\,\rho_{13}\sigma_1\sigma_3 v_1 v_2^2 v_3$$

$$- 4\,\rho_{23}\sigma_2\sigma_3 v_1^2 v_2 v_3 - \sigma_3^2 v_1^2 v_2^2 - \sigma_1^2 v_2^2 v_3^2$$

$$- \sigma_2^2 v_1^2 v_3^2 + [\sigma_2^2\sigma_3^2 + 2(\rho_{23}\sigma_2\sigma_3)^2] v_1^2$$

$$+ [\sigma_1^2\sigma_3^2 + 2(\rho_{13}\sigma_1\sigma_3)^2] v_2^2 + [\sigma_1^2\sigma_2^2 + 2(\rho_{12}\sigma_1\sigma_2)^2] v_3^2$$

$$+ [4\sigma_3^2\rho_{12}\sigma_1\sigma_2 + 8(\rho_{13}\sigma_1\sigma_3)(\rho_{23}\sigma_2\sigma_3)] v_1 v_2$$

$$+ [4\,\sigma_2^2\rho_{13}\sigma_1\sigma_3 + 8(\rho_{12}\sigma_1\sigma_2)(\rho_{23}\sigma_2\sigma_3)] v_1 v_3$$

$$+ [4\,\sigma_1^2\rho_{23}\sigma_2\sigma_3 + 8(\rho_{12}\sigma_1\sigma_2)(\rho_{13}\sigma_1\sigma_3)] v_2 v_3 \qquad \text{(F.100 Cont.)}$$

$$- [\sigma_1^2\sigma_2^2\sigma_3^2 + 2\,\sigma_1^2(\rho_{23}\sigma_2\sigma_3)^2 + 2\,\sigma_3^2(\rho_{12}\sigma_1\sigma_2)^2$$

$$+ 2\,\sigma_2^2(\rho_{13}\sigma_1\sigma_3)^2 + 8(\rho_{12}\sigma_1\sigma_2)(\rho_{13}\sigma_1\sigma_3)(\rho_{23}\sigma_2\sigma_3)]$$

Expansion of a Function in terms of the Polynomials

A function $F(v_1,v_2,v_3)$ can be expanded in terms of the trivariate Hermite polynomials as follows:

$$F(v_1,v_2,v_3) = \sum_{i,j,k=0}^{\infty} B_{i,j,k} H_{i,j,k}(v_1,v_2,v_3) \qquad \text{(F.101)}$$

or

$$F(v_1,v_2,v_3) = \sum_{i,j,k=0}^{\infty} C_{i,j,k}G_{i,j,k}(v_1,v_2,v_3) \tag{F.102}$$

where $B_{i,j,k} = (2\pi)^{-3/2}(\Delta)^{1/2}[i!j!k!]^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\exp\{-\frac{1}{2}\,\phi(v_1,v_2,v_3)\}]$

$$\cdot[F(v_1,v_2,v_3)G_{i,j,k}(v_1,v_2,v_3)]dv_1dv_2dv_3 \tag{F.103}$$

$$C_{i,j,k} = (2\pi)^{-3/2}(\Delta)^{1/2}[i!j!k!]^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\exp\{-\frac{1}{2}\,\phi(v_1,v_2,v_3)\}]$$

$$\cdot[F(v_1,v_2,v_3)\ H_{ijk}(v_1,v_2,v_3)]dv_1dv_2dv_3 \tag{F.104}$$

By a straightforward application of equations F.101 and F.103, one obtains the following useful equations:

$$v_1v_3 = \sigma_1\sigma_3\sigma_{13}H_{000} + \sigma_1^2\sigma_2\sigma_3(\rho_{23} + \rho_{12}\rho_{13})H_{110}$$

$$+ \sigma_1\sigma_2\sigma_3^2(\rho_{12} + \rho_{13}\rho_{23})H_{011} + \sigma_1^2\sigma_3^2\ (1 + \rho_{13}^2)H_{101} + \sigma_1^3\sigma_3\rho_{13}H_{200}$$

$$+ \sigma_1\sigma_2^2\sigma_3\rho_{12}\rho_{23}H_{020} + \sigma_1\sigma_3^3\rho_{13}H_{002} \tag{F.105}$$

$$v_2v_3 = \sigma_2\sigma_3\rho_{23}H_{000} + \sigma_1\sigma_2^2\sigma_3(\rho_{13} + \rho_{12}\rho_{23})H_{110}$$

$$+ \sigma_2^{\ 2}\sigma_3^{\ 2}(1 + \rho_{23}^{\ 2})H_{011} + \sigma_1\sigma_2\sigma_3^{\ 2}(\rho_{12} + \rho_{13}\rho_{23})H_{101}$$

$$+ \sigma_1^{\ 2}\sigma_2\sigma_3\rho_{12}\rho_{13}H_{200} + \sigma_2^{\ 3}\sigma_3\rho_{23}H_{020} + \sigma_2\sigma_3^{\ 3}\rho_{23}H_{002} \tag{F.106}$$

$$v_3^{\ 2} = \sigma_3^{\ 2}H_{000} + 2\ \sigma_1\sigma_2\sigma_3^{\ 2}\rho_{13}\rho_{23}H_{110}$$

$$+ 2\ \sigma_2\sigma_3^{\ 3}\rho_{23}H_{011} + 2\ \sigma_1\sigma_3^{\ 3}\ \rho_{13}H_{101}$$

$$+ 2\ \sigma_1^{\ 2}\sigma_3^{\ 2}\rho_{13}^{\ 2}H_{200} + 2\ \sigma_2^{\ 2}\sigma_3^{\ 2}\rho_{23}^{\ 2}H_{020} + 2\ \sigma_3^{\ 4}H_{002} \tag{F.107}$$

One can also expand a function $F(v_1,v_2,v_3)$ in a series of terms of the form $\exp\{-\frac{1}{2}\phi(v_1v_2v_3)\}H_{i,j,k}(v_1,v_2,v_3)$ as follows

$$F(v_1,v_2,v_3) = \sum_{i,j,k=0}^{\infty} D_{i,j,k} \exp\{-\tfrac{1}{2}\phi(v_1,v_2,v_3)\}H_{i,j,k}(v_1,v_2,v_3) \tag{F.108}$$

where

$$D_{i,j,k} = (2\pi)^{-3/2}\ (\Delta)^{1/2}\ [i!j!k!]^{-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}$$

$$F(v_1,v_2,v_3)\ G_{i,j,k}(v_1,v_2,v_3)\ dv_1dv_2dv_3 \tag{F.109}$$

and this type of expansion would be a trivariate Gram-Charlier series expansion.

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*Abbreviations herein follow form used by American Geophysical Union.

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VITA

William Battle Moseley was born October 8, 1942 in Savannah, Georgia. He is the son of William Belvin Moseley (deceased) and the present Mrs. Phillip C. De Bonis. On March 25, 1961 he was married to Sandra Lee Hosti of Savannah Beach, Georgia. He has three children; Sandra Michelle, Theresa Clair, and Kimberly Ann.

He attended Benedictine Military High School in Savannah, Georgia. In 1964 he received the degree of Bachelor of Science in Physics from the Georgia Institute of Technology and was awarded the degree of Master of Science in Physics from the same school in 1966.

The writer began work as a graduate research assistant in 1964 with the Space Science Laboratory at Georgia Tech. He has published a paper on part of his thesis which is indicated as reference 16 in the Bibliography.

While in college, the writer was a member of Chi Phi Social Fraternity. He was also a member of Tau Beta Pi and Sigma Pi Sigma honorary societies. He is a member of the American Geophysical Union.